TILTING AND COTILTING MODULES OVER CONCEALED CANONICAL ALGEBRAS

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Abstract. We study infinite dimensional tilting modules over a concealed canonical algebra of domestic or tubular type. In the domestic case, such tilting modules are constructed by using the technique of universal localization, and they can be interpreted in terms of Gabriel localizations of the corresponding category of quasi-coherent sheaves over a noncommutative curve of genus zero. In the tubular case, we have to distinguish between tilting modules of rational and irrational slope. For rational slope the situation is analogous to the domestic case. In contrast, for any irrational slope, there is just one tilting module of that slope up to equivalence. We also provide a dual description of infinite dimensional cotilting modules and a classification result for the indecomposable pure-injective modules.

1. Introduction

Infinite dimensional modules over canonical algebras have been studied by several authors, see e.g. [40, 39, 27, 28]. Particular attention has been devoted to the problem of classifying the pure-injective modules over canonical algebras of tubular type [41, 27, 28]. In this paper, we approach the problem from the viewpoint of tilting theory. Using the methods developed in [6] for tame hereditary algebras, we study the infinite dimensional tilting and cotilting modules over a concealed canonical algebra Λ of domestic or tubular type. The knowledge of these modules allows to obtain classification results for the indecomposable pure-injective Λ-modules. It further enables us to reinterpret the classification from [3] of the large quasi-coherent tilting sheaves over a noncommutative curve of genus zero in terms of modules over a derived equivalent algebra.

As in the hereditary case, the technique of universal localization due to Cohn and Schofield plays a fundamental role in the construction of tilting modules. Indeed, if $\mathbf{t} = \bigcup_{x\in\mathbb{X}} \mathcal{U}_x$ is a stable, sincere, separating tubular family yielding a trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ of the finite dimensional indecomposable Λ-modules, then any universal localization $\Lambda \to \Lambda_{\mathcal{U}}$ at a set $\mathcal{U}$ of modules in $\mathbf{t}$ is injective and gives rise to a tilting module $\Lambda_{\mathcal{U}} \oplus \Lambda_{\mathcal{U}}/\Lambda$.

If Λ has only homogeneous tubes, we obtain all but one infinite dimensional tilting Λ-modules in this way (up to equivalence). The missing one is called Lukas tilting module, and it generates the class $\mathcal{B}$ of all modules without non-zero maps to the modules in $\mathbf{p}$. More generally, if Λ has domestic representation type, large tilting modules can have a more complicated shape involving also finite dimensional summands, but the infinite dimensional part can still be described in terms of universal localizations $\Lambda_{\mathcal{U}}$ and Lukas tilting modules over such $\Lambda_{\mathcal{U}}$. We thus obtain a classification of the large tilting, and by duality, of the large cotilting modules, which is completely analogous to the tame hereditary case treated in [6, 14], see Theorem 6.2. In Corollaries 5.7 and 5.8 we also see that large tilting modules correspond to Gabriel localizations of the category of quasi-coherent sheaves $\operatorname{Qcoh}\mathbb{X}$ over the noncommutative curve of genus zero $\mathbb{X}$ corresponding to Λ.

Let us remark that in the domestic case all large tilting or cotilting modules are located in the "central part" of Mod-Λ, that is, given the trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$, they admit neither maps to $\mathbf{p}$ nor maps from $\mathbf{q}$. In other words, they belong to the intersection $\mathcal{M} = \mathcal{B} \cap \mathcal{C}$ of the torsion class $\mathcal{B}$ of the torsion pair $(\mathcal{B}, \mathcal{P})$ in Mod-Λ generated by $\mathbf{p}$ with the torsion-free class $\mathcal{C}$ of the torsion pair $(\mathcal{Q}, \mathcal{C})$ cogenerated by $\mathbf{q}$.

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In the tubular case, the AR-quiver consists of a preprojective component $\mathbf{p}_0$, a preinjective component $\mathbf{q}_\infty$, two non-stable tubular families $\mathbf{t}_0$ and $\mathbf{t}_\infty$, and a countable number of stable, sincere, separating tubular families $\mathbf{t}_w, w \in \mathbb{Q}^+$, giving rise to trisections $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$ of the finite dimensional indecomposable Λ-modules. So, one can construct a class $\mathcal{M}_w = \mathcal{B}_w \cap \mathcal{C}_w$ as above for every $w \in \mathbb{Q}^+$. For irrational $w \in \mathbb{R}^+$, one divides the finite dimensional indecomposable Λ-modules into the class $\mathbf{p}_w$ of all modules that belong to $\mathbf{p}_0$ or to one of the tubular families $\mathbf{t}_v$ with $v < w$, and the class $\mathbf{q}_w$ given by the remaining modules. The corresponding torsion class $\mathcal{B}_w$ and torsion-free class $\mathcal{C}_w$ have an intersection $\mathcal{M}_w$ which will only contain infinite dimensional modules.

Following [39], we say that the modules in $\mathcal{M}_w$ have slope w. This yields a notion of slope which is a module-theoretic counterpart of the notion of slope for sheaves over a noncommutative curve of genus zero, and which can be defined also for infinite dimensional modules. Indeed, it is shown in [39] that every indecomposable module (of finite or infinite dimension) has a slope.

For rational w, the classification of large tilting and cotilting modules of slope w is completely analogous to the domestic case. In contrast, for irrational w, there are just one tilting module $\mathbf{L}_w$ and one cotilting module $\mathbf{W}_w$ of that slope. Moreover, the modules of slope w are precisely the pure-epimorphic images of direct sums of copies of $\mathbf{L}_w$. Dually, they can be described as the pure submodules of products of copies of $\mathbf{W}_w$. In particular, every indecomposable pure-injective module of slope w belongs to $\operatorname{Prod} \mathbf{W}_w$. Combining this with results from [27], we obtain a classification of the indecomposable pure-injective Λ-modules in Theorem 6.7.

Furthermore, in Theorem 6.10, we show that every tilting module which does not belong to the leftmost part of Mod-Λ determined by $\mathbf{p}_0 \cup \mathbf{t}_0$, nor to the rightmost part determined by $\mathbf{t}_\infty \cup \mathbf{q}_\infty$, has a slope. This yields a classification of the tilting and cotilting modules in the "central part" of Mod-Λ.

Finally, let us turn to the category $\operatorname{Qcoh} \mathbb{X}$ of quasi-coherent sheaves over a noncommutative curve of genus zero $\mathbb{X}$. A classification of the large tilting sheaves, with a self-contained proof inside the hereditary category $\operatorname{Qcoh} \mathbb{X}$, is given in [3]. Here we illustrate the interplay between tilting sheaves and tilting modules over a derived equivalent concealed canonical algebra. More precisely, since by [3, Lem. 7.10] every tilting sheaf $\widehat{T}$ in $\operatorname{Qcoh} \mathbb{X}$ is generated by a suitable tilting bundle T_{cc} in $\operatorname{coh} \mathbb{X}$, we can regard $\operatorname{Qcoh} \mathbb{X}$ as the heart of a certain t-structure in the derived category of the algebra $\Lambda = \operatorname{End} T_{cc}$ and $\widehat{T}$ as a tilting module in the "central part" of Mod-Λ. This allows to recover the classification of large tilting sheaves from [3], see Theorem 6.11. In particular, this shows that also the tilting sheaves of rational slope can be described in terms of universal localizations and Lukas tilting modules.

The paper is organized as follows. After some preliminaries in Sections 2 and 3, we discuss the construction of tilting modules via universal localization in Section 4. Gabriel localizations of $\operatorname{Qcoh} \mathbb{X}$ are treated in Section 5. Section 6 is devoted to the classification results mentioned above.

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2. Preliminaries

Throughout this section, let $\mathcal{B}$ be a Grothendieck category. Given a class of objects $\mathcal{M} \subset \mathcal{B}$, we denote

$$\mathcal{M}^o = \{B \in \mathcal{B} \mid \operatorname{Hom}_{\mathcal{A}}(M, B) = 0 \text{ for all } M \in \mathcal{M}\}$$

$$\mathcal{M}^\perp = \{B \in \mathcal{B} \mid \operatorname{Ext}^1_{\mathcal{A}}(M, B) = 0 \text{ for all } M \in \mathcal{M}\}$$

The classes ${}^o\mathcal{M}$ and ${}^\perp\mathcal{M}$ are defined dually. If $\mathcal{M}$ consists of a single object M, we write M^o, $M^\perp$, etc.

2.1. Torsion pairs. Recall that a pair of classes $(\mathcal{T},\mathcal{F})$ in $\mathcal{B}$ is a *torsion pair* if $\mathcal{T} = {}^o\mathcal{F}$, and $\mathcal{F} = \mathcal{T}^o$. Every class $\mathcal{M}$ of objects in $\mathcal{B}$ *generates* a torsion pair $(\mathcal{T},\mathcal{F})$ by setting $\mathcal{F} = \mathcal{M}^o$ and $\mathcal{T} = {}^o(\mathcal{M}^o)$. Similarly, the torsion pair *cogenerated by* $\mathcal{M}$ is given by $\mathcal{T} = {}^o\mathcal{M}$ and $\mathcal{F} = ({}^o\mathcal{M})^o$. We say that a torsion pair $(\mathcal{T},\mathcal{F})$ is *split* if every short exact sequence $0 \to T \to M \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ splits. Finally, a torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{B}$ is *faithful* (or cotilting in the terminology of [26, 45]) if the torsionfree class $\mathcal{F}$ generates the category $\mathcal{B}$. Similarly, $(\mathcal{T},\mathcal{F})$ is *cofaithful* (or tilting in the terminology of [26, 45]) if the torsion class $\mathcal{T}$ cogenerates $\mathcal{B}$.

2.2. Gabriel localization. Torsion pairs play an important role in connection with the localization theory developed by Gabriel, see [22, 23, 38]. First of all, recall that a subcategory $\mathfrak{S}$ of $\mathcal{B}$ is a *Serre subcategory* if for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{B}$ we have $B \in \mathfrak{S}$ if and only if $A, C \in \mathfrak{S}$. We can then form the quotient category $\mathcal{B}/\mathfrak{S}$ with the canonical quotient functor $q : \mathcal{B} \to \mathcal{B}/\mathfrak{S}$. A torsion pair $(\mathcal{T},\mathcal{F})$ in $\mathcal{B}$ is said to be *hereditary* if the torsion class $\mathcal{T}$ is closed under subobjects, or in other words, $\mathcal{T}$ is a Serre subcategory of $\mathcal{B}$. If $\mathcal{B}$ has enough injectives, then $\mathcal{T}$ is even a *localizing* subcategory of $\mathcal{B}$, i.e. the quotient functor $q : \mathcal{B} \to \mathcal{B}/\mathcal{T}$ has a right adjoint. More details will be given in Theorem 5.6.

2.3. Tilting and cotilting objects. An object V of $\mathcal{B}$ is *tilting* if the category $\operatorname{Gen} V$ of V-generated objects equals $V^\perp$. Then

$$(\operatorname{Gen} V, V^o)$$

is a cofaithful torsion pair in $\mathcal{B}$, and $\operatorname{Gen} V$ is called a *tilting class*.
As shown in [16, 2.1 and 2.2], the equality $\operatorname{Gen} V = V^\perp$ is equivalent to the three conditions

(T1) $\operatorname{pdim} V \leq 1$, that is, the functor $\operatorname{Ext}^2_{\mathcal{B}}(V, -) = 0$,
(T2) $\operatorname{Ext}^1_{\mathcal{B}}(V, V^{(\alpha)}) = 0$ for all cardinals α,
(T3) an object $B \in \mathcal{B}$ is zero whenever it satisfies $\operatorname{Hom}_{\mathcal{B}}(V, B) = \operatorname{Ext}^1_{\mathcal{B}}(V, B) = 0$.

When $\mathcal{B} = \text{Mod-}\Lambda$ for some ring Λ, condition (T3) can be rephrased as follows:

(T3') There is a short exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ where $T_0, T_1 \in \operatorname{Add} T$.

Cotilting objects and *cotilting classes* in $\mathcal{B}$ are defined dually. In particular, a module W is cotilting if the category $\operatorname{Cogen} W$ of W-cogenerated objects equals ${}^\perp W$, or equivalently, C has the dual properties (C1)-(C3). Of course, the torsion pair

$$({}^oW, \operatorname{Cogen} W)$$

is then a faithful torsion pair in Mod-Λ.

We will discuss classification of tilting and cotilting modules up to equivalence. Hereby, we say that two tilting modules T, T' are *equivalent* if they induce the same tilting class $\operatorname{Gen} T = \operatorname{Gen} T'$, or equivalently, if they have the same additive closure $\operatorname{Add} T = \operatorname{Add} T'$. Similarly, two cotilting modules W, W' are *equivalent* if they induce the same cotilting class $\operatorname{Cogen} W = \operatorname{Cogen} W'$, or equivalently, $\operatorname{Prod} W = \operatorname{Prod} W'$.

When Λ is a left noetherian ring with a fixed duality D (for example $D = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, or $D = \operatorname{Hom}_K(-, K)$ in case Λ is a finite dimensional algebra over a field K), tilting and cotilting modules are related by the following result.

Theorem 2.1 ([10, 4])**.** *If Λ is a left noetherian ring, there is a bijection between*

(1) *equivalence classes of tilting modules in* Mod-Λ*,*
(2) *equivalence classes of cotilting modules in* Λ-Mod*,*
(3) *resolving subcategories of* mod-Λ *consisting of modules of projective dimension* ≤ 1.

The bijection above assigns to a tilting module T the cotilting module $D(T)$, and the resolving subcategory $\mathcal{S} = {}^{\perp}(\operatorname{Gen} T) \cap \text{mod-}\Lambda$. Conversely, a resolving subcategory $\mathcal{S}$ is mapped to the tilting class $\mathcal{S}^{\perp}$ and the cotilting class $\mathcal{S}^{\intercal} = \{B \in \Lambda\text{-Mod} \mid \operatorname{Tor}_1^{\Lambda}(S, B) = 0$ for all $S \in \mathcal{S}\}$.

Hereby, a subcategory $\mathcal{S} \subset \text{mod-}\Lambda$ (or of Mod-Λ) is said to be *resolving* if it is closed under direct summands, extensions, and kernels of epimorphisms, and it contains Λ (or all projective modules, respectively).

(Co)tilting modules that are not equivalent to a finitely generated (co)tilting module will be called *large*.

2.4. The heart. Let $(\mathcal{Q}, \mathcal{C})$ be a torsion pair in $\mathcal{B}$. According to [26, 45], the classes

$$\mathcal{D}^{\leq 0} = \{X^{\cdot} \in \mathcal{D}(\mathcal{B}) \mid H^0(X^{\cdot}) \in \mathcal{Q}, H^i(X^{\cdot}) = 0 \text{ for } i > 0\},$$

$$\mathcal{D}^{\geq 0} = \{X^{\cdot} \in \mathcal{D}(\mathcal{B}) \mid H^{-1}(X^{\cdot}) \in \mathcal{C}, H^i(X^{\cdot}) = 0 \text{ for } i < -1\}$$

form a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in the derived category $\mathcal{D}(\mathcal{B})$, called the *t-structure induced by* $(\mathcal{Q}, \mathcal{C})$. Its *heart*

$$\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$$

is always an abelian category [12] whose exact structure is given by the triangles of $\mathcal{D}(\mathcal{B})$. For any two objects $X, Z \in \mathcal{A}$ there are functorial isomorphisms

$$\operatorname{Ext}^i_{\mathcal{A}}(X, Z) \cong \operatorname{Hom}_{\mathcal{D}(\mathcal{B})}(X, Z[i]) \text{ for } i = 0, 1.$$

Moreover, $(\mathcal{C}[1], \mathcal{Q})$ is a torsion pair in $\mathcal{A}$ by [26, I.2.2].

From now on, we assume that $(\mathcal{Q}, \mathcal{C})$ is a faithful torsion pair and $\mathcal{A}$ is the heart of the corresponding t-structure in $\mathcal{D}(\mathcal{B})$. Then there is a triangle equivalence between $\mathcal{D}(\mathcal{B})$ and $D(\mathcal{A})$, see e.g. [45, 3.12]. We record the following facts for later reference.

Lemma 2.2. (1) *For every $X \in \mathcal{A}$ there are objects $Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$ with a canonical sequence*

$$0 \to Y[1] \to X \to Q \to 0.$$

(2) *The following statements hold true for $C, Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$.*
   (a) $\operatorname{Hom}_{\mathcal{A}}(Q, C[1]) \cong \operatorname{Ext}^1_{\mathcal{B}}(Q, C)$,
   (b) $\operatorname{Ext}^1_{\mathcal{A}}(Q, C[1]) \cong \operatorname{Ext}^2_{\mathcal{B}}(Q, C)$,
   (c) $\operatorname{Hom}_{\mathcal{A}}(Y[1], C[1]) \cong \operatorname{Hom}_{\mathcal{B}}(Y, C)$,
   (d) $\operatorname{Ext}^1_{\mathcal{A}}(Y[1], C[1]) \cong \operatorname{Ext}^1_{\mathcal{B}}(Y, C)$,
   (e) $\operatorname{Ext}^1_{\mathcal{A}}(Y[1], Q) \cong \operatorname{Hom}_{\mathcal{B}}(Y, Q)$.

Proof. is left to the reader. □

As shown in [45, 5.2], the heart $\mathcal{A}$ is a *hereditary* abelian category, that is, $\operatorname{Ext}^2_{\mathcal{A}}(-,-) = 0$, if and only if the torsion pair $(\mathcal{Q}, \mathcal{C})$ is split and all objects in $\mathcal{C}$ have projective dimension at most one. In this case, the (co)tilting objects in $\mathcal{A}$ and $\mathcal{B}$ are closely related.

Lemma 2.3. *Assume that $\mathcal{A}$ is hereditary.*
1. *An object $C \in \mathcal{B}$ of injective dimension at most one that belongs to $\mathcal{C}$ satisfies conditions (C2) and (C3) in $\mathcal{B}$ if and only if so does $C[1]$ in $\mathcal{A}$.*
2. *An object $T \in \mathcal{B}$ that belongs to $\mathcal{C}$ satisfies conditions (T2) and (T3) in $\mathcal{B}$ if and only if so does $T[1]$ in $\mathcal{A}$.*

Proof. It follows from Lemma 2.2(d) that C satisfies condition (C2) in $\mathcal{B}$ if and only if so does $C[1]$ in $\mathcal{A}$, and similarly T satisfies condition (T2) in $\mathcal{B}$ if and only if so does $T[1]$ in $\mathcal{A}$.

Assume now that C satisfies (C3), and let $X \in \mathcal{A}$ be an object satisfying $\operatorname{Hom}_{\mathcal{A}}(X, C[1]) = \operatorname{Ext}^1_{\mathcal{A}}(X, C[1]) = 0$. From the canonical sequence

$$0 \to Y[1] \to X \to Q \to 0$$

with $Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$ we obtain a long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Q, C[1]) \to 0 \to \mathrm{Hom}_{\mathcal{A}}(Y[1], C[1]) \to \mathrm{Ext}^1_{\mathcal{A}}(Q, C[1]) \to 0 \to \mathrm{Ext}^1_{\mathcal{A}}(Y[1], C[1]) \to 0.$$

We infer from Lemma 2.2(a) that $\mathrm{Ext}^1_{\mathcal{B}}(Q, C) = 0$, and as $\mathrm{Hom}_{\mathcal{B}}(Q, C) = 0$ by assumption, we conclude $Q = 0$. Moreover, it follows from Lemma 2.2(b) and (c) that $\mathrm{Hom}_{\mathcal{B}}(Y, C) \cong \mathrm{Ext}^1_{\mathcal{A}}(Q, C[1]) \cong \mathrm{Ext}^2_{\mathcal{B}}(Q, C) = 0$ as C has injective dimension at most one. Since we also have $\mathrm{Ext}^1_{\mathcal{B}}(Y, C) = 0$ by Lemma 2.2(d), we conclude $Y = 0$, and so $X = 0$. This shows that $C[1]$ satisfies condition (C3).
Conversely, assume that $C[1]$ satisfies condition (C3), and let $B \in \mathcal{B}$ be an object satisfying $\mathrm{Hom}_{\mathcal{B}}(B, C) = \mathrm{Ext}^1_{\mathcal{B}}(B, C) = 0$. Then $B = Y \oplus Q$ with $Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$. By Lemma 2.2(c) and (d) it follows that $\mathrm{Hom}_{\mathcal{A}}(Y[1], C[1]) = \mathrm{Ext}^1_{\mathcal{A}}(Y[1], C[1]) = 0$, hence $Y = 0$. Further, again from Lemma 2.2(b) we obtain $\mathrm{Ext}^1_{\mathcal{A}}(Q, C[1]) = 0$. Since $\mathrm{Hom}_{\mathcal{A}}(Q, C[1]) = 0$ by Lemma 2.2(a), we conclude $Q = 0$ and $B = 0$. So C satisfies (C3) as well.
The proof of statement (2) is dual. □

2.5. Hearts induced by cotilting modules. Assume now $\mathcal{B} = $ Mod-Λ for some ring Λ. Then, as shown in [16, Sections 3 and 4], the object $V = \Lambda[1] \in \mathcal{A}$ is a tilting object with $\mathrm{End}_{\mathcal{A}} V \cong \Lambda$, defining crosswise equivalences

$$H_V = \mathrm{Hom}_{\mathcal{A}}(V, -) : \mathcal{C}[1] = \mathrm{Gen}\, V \to \mathcal{C}, \quad H'_V = \mathrm{Ext}^1_{\mathcal{A}}(V, -) : \mathcal{Q} = V^o \to \mathcal{Q}$$

between the torsion class in $\mathcal{A}$ and the torsionfree class in Mod-Λ, and between the torsion-free class in $\mathcal{A}$ and the torsion class in Mod-Λ, respectively. In other words, $\mathbb{R}\,\mathrm{Hom}_{\mathcal{A}}(V, -)$ yields an equivalence $D(\mathcal{A}) \to \mathcal{D}(\text{Mod-}\Lambda)$.

Proposition 2.4. *Let $(\mathcal{Q}, \mathcal{C})$ be a faithful torsion pair in* Mod-Λ *with hereditary heart $\mathcal{A}$.*

1. *If $C \in \mathcal{C}$ is a cotilting Λ-module, then $\mathrm{Cogen}\, C[1] = {}^{\perp}C[1]$ is a torsionfree class in $\mathcal{A}$ consisting of the objects in $X \in \mathcal{A}$ for which $H_V(X) \in \mathrm{Cogen}\, C$. In particular, $\mathcal{Q}$ is always contained in $\mathrm{Cogen}\, C[1]$.*
2. *If $T \in \mathcal{C}$ is a tilting Λ-module, then $\mathrm{Gen}\, T[1] = T[1]^{\perp}$ is a torsion class in $\mathcal{A}$ consisting of the objects in $X \in \mathcal{A}$ for which $H_V(X) \in \mathrm{Gen}\, T$ and $H'_V(X) \in T^o$.*

Proof. Since $\mathcal{A}$ is hereditary, conditions (C1) and (T1) are always satisfied in $\mathcal{A}$.
(1) We know from Lemma 2.3 that $C[1]$ satisfies (C1)-(C3). Dualizing the proof of [16, 2.1], we infer that Cogen $C[1] = {}^{\perp}C[1]$ is a torsionfree class in $\mathcal{A}$. Let $X \in \mathcal{A}$. Taking again the canonical sequence $0 \to Y[1] \to X \to Q \to 0$ with $Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$ and recalling that $\mathrm{Ext}^1_{\mathcal{A}}(Q, C[1]) \cong \mathrm{Ext}^2_{\Lambda}(Q, C) = 0$ and $\mathrm{Ext}^1_{\mathcal{A}}(Y[1], C[1]) \cong \mathrm{Ext}^1_{\Lambda}(Y, C)$ by Lemma 2.2(b) and (d), we see that $X \in$ Cogen $C[1]$ if and only if $Y \in \mathrm{Cogen}\, C$. The claim now follows from the fact that $Y = H_V(X)$ and $H_V(Q) = 0$ for all $Q \in \mathcal{Q}$.
(2) Again we infer from Lemma 2.3 and the proof of [16, 2.1] that Gen $T[1] = T[1]^{\perp}$ is a torsion class in $\mathcal{A}$. Applying $\mathrm{Hom}_{\mathcal{A}}(T[1], -)$ to the canonical sequence $0 \to Y[1] \to X \to Q \to 0$ with $Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$, we obtain a long exact sequence $0 = \mathrm{Hom}_{\mathcal{A}}(T[1], Q) \to \mathrm{Ext}^1_{\mathcal{A}}(T[1], Y[1]) \to \mathrm{Ext}^1_{\mathcal{A}}(T[1], X) \to \mathrm{Ext}^1_{\mathcal{A}}(T[1], Q) \to 0$. By Lemma 2.2(d) and (e), we see that $\mathrm{Ext}^1_{\mathcal{A}}(T[1], X) = 0$ if and only if $Y \in \mathrm{Gen}\, T$ and $\mathrm{Hom}_{\Lambda}(T, Q) = 0$. The claim now follows from the fact that $Y = H_V(X)$ and $Q = H'_V(X)$. □

We will be particularly interested in the case when $\mathcal{A}$ is a *Grothendieck category.* It was shown in [17] that this happens if and only if there is a cotilting module W such that $\mathcal{C} = \mathrm{Cogen}\, W$ (and $\mathcal{Q} = {}^{o}W$). Then $W[1]$ is an injective cogenerator of $\mathcal{A}$. In some cases, $\mathcal{A}$ has also the following geometric interpretation.

Proposition 2.5. *Let Λ be a connected artin algebra, and let $(\mathcal{Q}, \mathcal{C})$ be a torsion pair in* Mod-Λ. *Suppose that*

- (i) *there is a Σ-pure-injective cotilting Λ-module W such that $\mathcal{C} = \mathrm{Cogen}\, W$,*
- (ii) *the torsion pair $(\mathcal{Q}, \mathcal{C})$ splits,*

(iii) $\Lambda \in \mathcal{C}$ *and* $D({}_\Lambda\Lambda) \in \mathcal{Q}$,
(iv) $\mathcal{Q} \cap {}^\perp\mathcal{Q} = 0$,
(v) *all modules in* $\mathcal{C}$ *have projective dimension at most one.*

Then the heart $\mathcal{A}$ *of the corresponding t-structure in* $\mathcal{D}(\text{Mod-}\Lambda)$ *is equivalent to the category* $\operatorname{Qcoh}\mathbb{X}$ *of quasi-coherent sheaves over a noncommutative curve of genus zero* $\mathbb{X}$. *Hereby the category fp*$\mathcal{A}$ *of finitely presented objects corresponds to the category* $\operatorname{coh}\mathbb{X}$ *of coherent sheaves.*

Proof. First of all, the Σ-pure-injectivity of W yields by [18] that $\mathcal{A}$ is a locally noetherian Grothendieck category. Moreover, since the torsion pair $(\mathcal{Q}, \mathcal{C})$ splits and all modules in $\mathcal{C}$ have projective dimension at most one, it follows from [45, 5.2] that $\mathcal{A}$ is a hereditary category, that is, $\operatorname{Ext}^2_{\mathcal{A}}(-,-) = 0$.
Set $\mathcal{H} = \operatorname{fp}\mathcal{A}$. Then $\mathcal{H}$ is a noetherian hereditary category with tilting object $V = \Lambda[1]$. The objects of $\mathcal{H}$ are extensions of objects of the form $Y[1]$ with $Y \in \mathcal{C} \cap \text{mod-}\Lambda$ by objects $Q \in \mathcal{Q} \cap \text{mod-}\Lambda$. It is then clear that $\mathcal{H}$ is a connected, skeletally small abelian k-category for which all morphism and extension spaces are finite dimensional k-vector spaces.
We claim that $\mathcal{H}$ has no non-zero projective objects. Assume that $A \in \mathcal{H}$ is a non-zero projective object with canonical exact sequence $0 \to Y[1] \to A \to Q \to 0$. Then also $Y[1]$ is projective because $\mathcal{H}$ is hereditary. Since $\mathcal{Q}$ contains an injective cogenerator of Mod-Λ, the condition $\operatorname{Ext}^1_{\mathcal{H}}(Y[1], Q') \cong \operatorname{Hom}_\Lambda(Y, Q') = 0$ for all $Q' \in \mathcal{Q}$ implies that $Y[1] = 0$ and $A \cong Q \in \mathcal{Q}$. By condition (iv) there is a non-split short exact sequence $0 \to Q' \to B \xrightarrow{g} Q \to 0$ in Mod-Λ with all terms in $\mathcal{Q}$. We show that the sequence is also exact in the heart $\mathcal{A}$. To this end, we apply [24, pp.281] to compute the kernel and cokernel of $g : B \to Q$ viewed as a morphism in $\mathcal{A}$: if Z is the cone of g in $\mathcal{D}(\text{Mod-}\Lambda)$ and $K = \tau_{\leq -1}Z \to Z \to \tau_{\geq 0}Z \to K[1]$ is the canonical triangle where $\tau_{\leq -1}Z \in \mathcal{D}^{\leq -1}$ and $\tau_{\geq 0}Z \in \mathcal{D}^{\geq 0}$, then $\operatorname{Ker}_{\mathcal{A}}(g) = K[-1]$, and $\operatorname{Coker}_{\mathcal{A}}(g) = \tau_{\geq 0}Z$. But Z has homologies $Q' \in \mathcal{Q}$ in degree -1 and 0 elsewhere, thus $Z \cong K$, and g is an epimorphism in $\mathcal{A}$ with kernel Q'. So we have a non-split exact sequence in $\mathcal{A}$ ending at the projective object Q, a contradiction.
By the axiomatic description of $\operatorname{coh}\mathbb{X}$ given in [35, 2.5], we now conclude that $\mathcal{H} = \operatorname{coh}\mathbb{X}$ for a noncommutative curve of genus zero $\mathbb{X}$. Hereby $\mathbb{X}$ is obtained as an index set when decomposing the category $\mathcal{H}_0$ of indecomposable finite length objects of $\mathcal{H}$ into a family of connected uniserial length categories $\mathcal{H}_0 = \bigcup_{x \in \mathbb{X}} \mathcal{U}_x$. Finally $\mathcal{A} = \operatorname{Qcoh}\mathbb{X}$, cf. [22, Ch. VI] or [20, 5.4]. □

We can now improve Proposition 2.4 as follows.

Corollary 2.6. *Under the assumptions of Proposition 2.5 above, a* Λ*-module* $T \in \mathcal{C}$ *is a tilting module if and only if* $T[1]$ *is a tilting object in* $\mathcal{A}$. *In this case,* $\operatorname{Gen} T[1] = \{X \in \mathcal{A} \mid H_V(X) \in \operatorname{Gen} T$ *and* $H'_V(X) = 0\}$. *In particular,* $\operatorname{Gen} T[1]$ *is contained in* $\mathcal{C}[1]$.

Proof. Observe first that $T \in \mathcal{C}$ and $T[1] \in \mathcal{A}$ have projective dimension at most one. Moreover, both Mod-Λ and $\mathcal{A}$ have enough injectives. The first statement then follows immediately from Proposition 2.3.
By Proposition 2.4, $\operatorname{Gen} T[1]$ consists of the $X \in \mathcal{A}$ for which $H_V(X) \in \operatorname{Gen} T$ and $H'_V(X) \in T^o$, that is, $\operatorname{Hom}_\Lambda(T, H'_V(X)) = 0$. But $H'_V(X)$ is a module from $\mathcal{Q}$, and since $T \in \mathcal{C} \subset {}^\perp\mathcal{Q}$ by assumption (ii), we always have $\operatorname{Ext}^1_\Lambda(T, H'_V(X)) = 0$. By condition (T3) for the tilting module T we get that $H'_V(X) = 0$ whenever $X \in \operatorname{Gen} T[1]$. So $\operatorname{Gen} T[1] \subset \mathcal{C}[1]$. □

3. Concealed canonical algebras

3.1. The setup. *From now on* Λ denotes a finite dimensional, connected, concealed canonical algebra over a field k, for example a tame hereditary algebra, or a canonical algebra. By [36] concealed canonical algebras are precisely the finite dimensional algebras with a sincere stable separating tubular family $\mathbf{t} = \bigcup_{x \in \mathbb{X}} \mathcal{U}_x$ yielding a canonical trisection

$$(\mathbf{p}, \mathbf{t}, \mathbf{q})$$

of the category mod-Λ.
More precisely, $\mathbf{t}$ is a family of standard tubes $\mathcal{U}_x$ in the Auslander-Reiten quiver of Λ which is
- *sincere*: every simple module occurs as the composition factor of at least one module from $\mathbf{t}$;
- *stable*: it does not contain indecomposable projective or injective modules;
- *separating*: the indecomposable modules in mod-Λ that do not belong to $\mathbf{t}$ fall into two classes $\mathbf{p}$ and $\mathbf{q}$ such that $\mathrm{Hom}(\mathbf{q},\mathbf{p}) = \mathrm{Hom}(\mathbf{q},\mathbf{t}) = \mathrm{Hom}(\mathbf{t},\mathbf{p}) = 0$, and any homomorphism from a module in $\mathbf{p}$ to a module in $\mathbf{q}$ factors through any $\mathcal{U}_x$.

The modules in $\mathrm{add}\,\mathbf{t}$ form an exact abelian subcategory of mod-Λ in which all objects have finite length. The simple objects and the composition factors in this category will be called *simple regular* modules and *regular composition factors*. The set of all simple regular modules in a tube $\mathcal{U}_x$ is called the *clique* of $\mathcal{U}_x$. The order of the clique is the *rank* of $\mathcal{U}_x$. Notice that almost all tubes are *homogeneous*, i.e. of rank one.
Every simple regular module $S = S_1 \in \mathcal{U}_x$ determines a *ray* $\{S_n \mid n \in \mathbb{N}\}$ of $\mathcal{U}_x$, where S_n denotes the indecomposable object of regular length n with regular socle S. The direct limit of the modules on a ray $S_\infty = \varinjlim S_n$ is called *Prüfer module*, the *adic module* $S_{-\infty}$ is defined dually. Both are indecomposable, infinite dimensional, pure-injective modules. By abuse of terminology, we say that S_∞, or $S_{-\infty}$, is a Prüfer module, respectively an adic module, *from the tube* $\mathcal{U}_x$.
Given a tube $\mathcal{U}_x$ of rank $r > 1$ and a module $S_m \in \mathcal{U}_x$ of regular length $m < r$, we consider the full subquiver $\mathcal{W}_{S_m}$ of $\mathcal{U}_x$ which is isomorphic to the Auslander-Reiten-quiver $\Theta(m)$ of the linearly oriented quiver of type $\mathbb{A}_m$ with S_m corresponding to the projective-injective vertex of $\Theta(m)$. The set $\mathcal{W}_{S_m}$ is called a *wing* of $\mathcal{U}_x$ of size m with *vertex* S_m.

It is shown in [39, 3.1 and §10] that the class $\mathbf{q}$ generates a split torsion pair
$$(\mathrm{Gen}\,\mathbf{q}, \mathcal{C})$$
in Mod-Λ and that $\mathcal{C} = \mathrm{Cogen}\,\mathbf{W}$ for a cotilting module $\mathbf{W}$ which is the direct sum of of all Prüfer modules S_∞, where S runs through the isoclasses of all simple regular Λ-modules in $\mathbf{t}$, and an indecomposable infinite dimensional module G which has finite length over its endomorphism ring and is called the *generic* module. Note that in the tame hereditary case $\mathrm{Gen}\,\mathbf{q} = \mathrm{Add}\,\mathbf{q}$ and $\mathcal{C}$ is the largest cotilting class in Mod-Λ which is induced by a large cotilting module (cf.[6, §2]).

We consider the t-structure induced by the torsion pair $(\mathrm{Gen}\,\mathbf{q}, \mathcal{C})$ in $\mathcal{D}(\text{Mod-}\Lambda)$, and denote its heart by $\mathcal{A}$. We claim that $\mathcal{A}$ is equivalent to the category $\mathrm{Qcoh}\,\mathbb{X}$ of quasi-coherent sheaves over $\mathbb{X}$.

Indeed, $\mathbf{W}$ is a Σ-pure-injective cotilting module, and we infer as above that $\mathcal{A}$ is a hereditary locally noetherian Grothendieck category with injective cogenerator $\mathbf{W}[1]$, see also [39, 11.1]. The indecomposable injective objects in $\mathcal{A}$ are $G[1]$ and the objects $S_\infty[1]$ where S runs through the isoclasses of all simple regular Λ-modules. Notice that $S_\infty[1]$ is a uniserial object with socle $S[1]$, and $\mathcal{H}_0 = \mathbf{t}[1]$ is the category of indecomposable finite length objects in $\mathcal{A}$.

Of course $\Lambda \in \mathcal{C} = {}^\perp\mathbf{W}$, and $D(\Lambda_\Lambda) \in \mathrm{Gen}\mathbf{q}$ since $\mathbf{t}$ is stable. Further, all modules in $\mathcal{C}$ have projective dimension at most one by [39, 5.4]. Finally, ${}^\perp(\mathrm{Gen}\,\mathbf{q}) \subset {}^\perp\mathbf{q} = \mathcal{C}$, hence ${}^\perp(\mathrm{Gen}\,\mathbf{q}) \cap \mathrm{Gen}\,\mathbf{q} = 0$. So conditions (i) - (v) in Proposition 2.5 are satisfied, and we deduce that $\mathcal{A}$ is equivalent to the category of quasi-coherent sheaves over a noncommutative curve of genus zero, which coincides with $\mathbb{X}$ because $\mathcal{H}_0 = \mathbf{t}[1] = \bigcup_{x \in \mathbb{X}} \mathcal{U}_x[1]$.

The indecomposable finitely presented objects of infinite length in $\mathcal{A}$ form the class $\mathrm{vect}\,\mathbb{X} = \mathbf{q} \cup \mathbf{p}[1]$ of indecomposable *vector bundles*. Notice that $\mathbf{t}$ generates the torsion pair $(\mathrm{Gen}\,\mathbf{t}, \varinjlim \mathbf{p})$ in Mod-Λ with torsion-free class $\varinjlim \mathbf{p} = \mathrm{Cogen}\,G$ by [39, 3.5 and 6.6], and $\mathcal{H}_0$ generates a torsion pair $(\varinjlim \mathbf{t}[1], \varinjlim \mathrm{vect}\,\mathbb{X})$ in $\mathcal{A}$. We call a module or a sheaf *torsion*, respectively *torsion-free*, if it is torsion, respectively torsion-free, with respect to these torsion pairs. Finally, $S_\infty[1]$, $G[1]$, $S_{-\infty}[1]$ are called *Prüfer, generic, adic sheaves*, and *wings* in the tubular family $\mathcal{H}_0$ are defined in analogous way as above.

3.2. Representation type. According to [36, Theorem 7.1], a numerical invariant called genus determines the representation type of the algebra Λ, which can be *domestic, tubular* or *wild*. In the domestic case, Λ is *tame concealed*, i. e. it can be realized as endomorphism ring of a preprojective or preinjective tilting module over a finite dimensional tame hereditary algebra Λ'. The tubular case will be discussed in more detail in Section 6.

3.3. The Auslander-Reiten formula. Denote by $(\mathcal{D},\mathcal{R})$ the torsion pair cogenerated by $\mathbf{t}$. As shown in [39], it is a split torsion pair, and the module $\mathbf{W}$ considered in 3.1 is a tilting module whose tilting class is the class of *divisible modules* $\mathcal{D} = \operatorname{Gen}\mathbf{W}$. By [39, 10.1]

$$\mathcal{C}\cap\mathcal{D} = \operatorname{Add}\mathbf{W} = \operatorname{Prod}\mathbf{W}.$$

Lemma 3.1. [39, 5.4] *The modules in $\mathcal{C}$ have projective dimension at most one, the modules in $\mathcal{D}$ have injective dimension at most one.*

In particular, the modules in $\mathbf{p}$ have projective dimension at most one, while those in $\mathbf{q}$ have injective dimension at most one. We will frequently use the following version of the Auslander-Reiten formula without further reference.

Lemma 3.2. [46] *Let A, C be Λ-modules, and assume that A is finitely generated without non-zero projective summands.*

(1) *If* $\operatorname{pdim} A \le 1$, *then* $\operatorname{Hom}_\Lambda(C,\tau A) \cong D\operatorname{Ext}^1_\Lambda(A,C)$.
(2) *If* $\operatorname{idim}\tau A \le 1$, *then* $D\operatorname{Hom}_\Lambda(A,C) \cong \operatorname{Ext}^1_\Lambda(C,\tau A)$.

As a first application, we see that $\mathcal{D} = {}^o\mathbf{t} = \mathbf{t}^\perp$ and $\mathcal{C} = \mathbf{q}^o = {}^\perp\mathbf{q}$. Further, we consider the torsion pair $(\mathcal{B},\mathcal{P})$ in Mod-Λ cogenerated by the class $\mathbf{p}$. Then $\mathcal{B} = {}^o\mathbf{p} = \mathbf{p}^\perp$, and by Theorem 2.1 there is a tilting module $\mathbf{L}$ with tilting class $\operatorname{Gen}\mathbf{L} = \mathcal{B}$. By [30, 2.2], $\mathbf{L}$ has an infinite filtration by modules in $\mathbf{p}$, so in particular it is torsion-free and belongs to $\mathcal{C}$. We call it the *Lukas tilting module* as its construction in the hereditary case goes back to [37], cf. [30, 3.3].

Lemma 3.3. *Let X be a Prüfer module, or an adic module, or the generic module. Further, let $P\in\mathcal{P}$ and $Q\in\mathcal{Q}$ be non-zero modules. Then* $\operatorname{Ext}^1_\Lambda(Q,X)\neq 0$ *and* $\operatorname{Ext}^1_\Lambda(X,P)\neq 0$.

Proof. First of all, one shows as in [14, 2.5] that $\operatorname{Ext}^1_\Lambda(Q,X)\neq 0$ when $Q\in\mathbf{q}$, and $\operatorname{Ext}^1_\Lambda(X,P)\neq 0$ when $P\in\mathbf{p}$.
Assume now $\operatorname{Ext}^1_\Lambda(Q,X)=0$. Notice that X has injective and projective dimension at most one. When X is a Prüfer module or the generic module, this follows from Lemma 3.1, and for adic modules it follows by duality. Hence $\operatorname{Ext}^1_\Lambda(Q',X)=0$ for all submodules Q' of Q. So Q cannot have submodules in $\mathbf{q}$ and therefore all its finitely generated submodules lie in $\mathcal{C}$. But then $Q\in\varinjlim\mathcal{C}=\mathcal{C}$, a contradiction.
For the second statement we proceed dually. If $\operatorname{Ext}^1_\Lambda(X,P)=0$, it follows that $\operatorname{Ext}^1_\Lambda(X,P')=0$ for all quotients P' of P. So all finitely generated factor modules of P lie in $\mathcal{B}$. Since every module can be purely embedded in the direct product of all its finitely generated factor modules, see [21, 2.2. Ex 3], and $\mathcal{B}$ is closed under direct products and pure submodules, we infer $P\in\mathcal{B}$, a contradiction. □

3.4. Purity. The objects of $\operatorname{fp}\mathcal{A} = \operatorname{coh}\mathbb{X}$ are pure-injective. Indeed, if $\varepsilon: 0\to A\to B\to C\to 0$ is a pure exact sequence in $\mathcal{A}$ and $X\in\mathcal{A}$ is finitely presented, then $\operatorname{Hom}_\mathcal{A}(\tau^- X,\varepsilon)$ is exact. Since $\operatorname{Ext}^2_\mathcal{A}(-,-)$ vanishes, this amounts to exactness of $\operatorname{Ext}^1_\mathcal{A}(\tau^- X,\varepsilon)$, which in turn is equivalent to exactness of $D\operatorname{Hom}_\mathcal{A}(\varepsilon,X)$ by Serre duality. But this means that $\operatorname{Hom}_\mathcal{A}(\varepsilon,X)$ is exact, which gives the claim.
Furthermore, a module $C\in\mathcal{C}$ is pure-injective if and only if $C[1]$ is a pure-injective object in $\mathcal{A}$. This follows from the following criterion by Jensen and Lenzing.

Lemma 3.4. [38, Theorem 5.4] *An object A in a locally noetherian category is pure-injective if and only if the summation map $A^{(I)}\to A$ factors through the canonical embedding $A^{(I)}\to A^I$ for every set I.*

Proposition 3.5. *Assume Λ has domestic representation type.*

(1) *The indecomposable pure-injective objects in $\mathcal{A} = \operatorname{Qcoh}\mathbb{X}$ are precisely the indecomposable coherent objects, the Prüfer sheaves, the adic sheaves, and the generic sheaf.*
(2) *The indecomposable pure-injective Λ-modules are precisely the finite dimensional indecomposable modules, the Prüfer modules, the adic modules and the generic module.*

Proof. (1) By assumption on Λ, there is a finitely presented tilting object $V \in \mathcal{A} = \operatorname{Qcoh}\mathbb{X}$ inducing a derived equivalence $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\text{Mod-}\Lambda')$ for a tame hereditary algebra Λ'. Denote by $\mathbf{q}'$ the preinjective component of Λ', and by $(\operatorname{Add}\mathbf{q}', \mathcal{C}')$ the corresponding split torsion pair in Mod-Λ'. Then $(\mathcal{C}'[1], \operatorname{Add}\mathbf{q}')$ is a split torsion pair in $\mathcal{A}$, because the corresponding heart Mod-Λ' is a hereditary category. So, every indecomposable non-coherent pure-injective object $A \in \mathcal{A}$ belongs to $\mathcal{C}'[1]$, and the claim follows from the discussion above and the well-known classification of pure-injective modules over tame hereditary algebras (see e.g. [21]).
(2) We first show that $\operatorname{Gen}\mathbf{q} = \operatorname{Add}\mathbf{q}$, as in the hereditary case. Recall that $\operatorname{Gen}\mathbf{q}$ is the direct limit closure of $\mathbf{q}$ by [20] or [25, 4.5.2]. So every $Q \in \operatorname{Gen}\mathbf{q}$ has the form $Q = \varinjlim Q_i$ for a suitable system $(Q_i)_{i\in I}$ from $\mathbf{q}$, and there is a pure-exact sequence $\varepsilon : 0 \to K \to \bigoplus_{i\in I} Q_i \to Q \to 0$. Let $\overline{K} \in \mathcal{C}$ be the torsion-free part of K, which is a direct summand as the torsion pair $(\operatorname{Gen}\mathbf{q}, \mathcal{C})$ splits. Then, since $\mathcal{C}$ is definable, the pure-injective envelope of $\overline{K}$ lies in $\mathcal{C}$ and factors through the pure-monomorphism $K \to \bigoplus_{i\in I} Q_i$. This shows that $\overline{K} = 0$ and ε has all terms in $\operatorname{Gen}\mathbf{q}$. As in the proof of Proposition 2.5, we infer that ε is exact in $\mathcal{A}$, and one easily checks that it is even pure-exact. Recall that the tilting object V induces a functor $H_V = \operatorname{Hom}_{\mathcal{A}}(V, -)$ with kernel $\operatorname{Add}\mathbf{q}'$. Now $H_V(\varepsilon)$ is exact, thus $H_V(Q) = 0$, showing that $Q \in \operatorname{Add}\mathbf{q}'$ is a direct sum of coherent objects in $\mathcal{A}$. Viewed as a Λ-module, Q is then a direct sum of finitely presented modules in $\operatorname{Gen}\mathbf{q}$, thus $Q \in \operatorname{Add}\mathbf{q}$.
Now we infer that every indecomposable infinite dimensional pure-injective module X must belong to $\mathcal{C}$, hence $X[1]$ is an indecomposable pure-injective object in $\mathcal{A}$, and the claim follows from (1). □

4. Universal localization.

In this section, we review the technique of universal localization developed by Cohn and Schofield, which is needed for the construction of tilting modules.

Theorem 4.1 ([42]). *Let R be a ring. For any set of morphisms Σ between finitely generated projective right R-modules there is a ring homomorphism $\lambda\colon R \to R_\Sigma$ such that*

(1) *λ is* Σ-inverting: *if $\alpha\colon P \to Q$ belongs to Σ, then the R_Σ-homomorphism $\alpha \otimes_R 1_{R_\Sigma} : P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism.*
(2) *λ is* universal *with respect to (1): any further Σ-inverting ring homomorphism $\lambda' : R \to R'$ factors uniquely through λ.*

The homomorphism $\lambda\colon R \to R_\Sigma$ is a ring epimorphism with $\operatorname{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$, called the universal localization *of R at Σ.*

Let now $\mathcal{E} \subset \text{mod-}\Lambda$ be a set of modules of projective dimension one. For each $E \in \mathcal{E}$, we fix a projective resolution $0 \to P \overset{\alpha_E}{\to} Q \to E \to 0$ in mod-Λ, and we set $\Sigma = \{\alpha_E \mid E \in \mathcal{E}\}$. We denote by $\Lambda_{\mathcal{E}}$ the universal localization of Λ at Σ, which does not depend on the chosen class Σ by [15, Theorem 0.6.2].

Theorem 4.2. *Let $\mathcal{U}$ be a set of simple regular modules. Then there is a short exact sequence*

$$0 \to \Lambda \overset{\lambda}{\to} \Lambda_{\mathcal{U}} \to \Lambda_{\mathcal{U}}/\Lambda \to 0$$

where

(1) *λ is a homological ring epimorphism, i.e. $\operatorname{Tor}_i^\Lambda(\Lambda_{\mathcal{U}}, \Lambda_{\mathcal{U}}) = 0$ for all $i > 0$,*
(2) *$\mathcal{U}^\wedge = \mathcal{U}^o \cap \mathcal{U}^\perp$ is the essential image of the restriction functor $\lambda_* : \text{Mod-}\Lambda_{\mathcal{U}} \to \text{Mod-}\Lambda$,*

(3) $\Lambda_{\mathcal{U}}/\Lambda$ *is a directed union of finite extensions of modules in* $\mathcal{U}$,
(4) $T_{\mathcal{U}} = \Lambda_{\mathcal{U}} \oplus \Lambda_{\mathcal{U}}/\Lambda$ *is a tilting module with tilting class* $\operatorname{Gen} T_{\mathcal{U}} = \mathcal{U}^{\perp}$.

Proof. Let $\mathcal{E}$ be the extension closure of $\mathcal{U}$. First of all, note that $\Lambda_{\mathcal{U}}$ coincides with $\Lambda_{\mathcal{E}}$, and $\mathcal{U}^o = \mathcal{E}^o$, $\mathcal{U}^{\perp} = \mathcal{E}^{\perp}$, $\mathcal{U}^{\wedge} = \mathcal{E}^{\wedge}$, cf. [6, 1.7]. Further, $\mathcal{E}$ is a class of finitely presented modules of projective dimension one which is closed under images, kernels, cokernels, and extensions, such $\Lambda \in \mathcal{E}^o$, so it is a well-placed subcategory of bound modules in the terminology of [43, 44]. It then follows from [43, 5.5 and 5.7] that λ is an injective homological epimorphism. Statement (2) is shown in [1, 2.7]. Moreover, since Λ is noetherian, for any finitely generated module M, the torsion submodule of M with respect to the torsion pair generated by $\mathcal{E}$ is finitely generated. Then one shows as in [44, 2.6] that $\Lambda_{\mathcal{E}}$ is a directed union of modules M_t containing Λ such that $M_t/\Lambda \in \mathcal{E}$, and statement (3) is an immediate consequence. Finally, since Λ is perfect, the class of modules of projective dimension at most one is closed under direct limits. We infer that $\Lambda_{\mathcal{U}}/\Lambda$ and $\Lambda_{\mathcal{U}}$ have projective dimension at most one, and [5, 3.10 and 4.12] yield statement (4). □

Let us describe the left adjoint $\lambda^* = - \otimes_{\Lambda} \Lambda_{\mathcal{U}} : \text{Mod-}\Lambda \to \text{Mod-}\Lambda_{\mathcal{U}}$ of the restriction functor $\lambda_* : \text{Mod-}\Lambda_{\mathcal{U}} \to \text{Mod-}\Lambda$.

Lemma 4.3. *(cf.* [6, 1.7]*) Let* $\mathcal{U}$ *be a set of simple regular modules, let* $(\mathcal{T}, \mathcal{F})$ *be the torsion pair generated by* $\mathcal{U}$*, and let* t *be the associated torsion radical.*
(1) $\mathcal{T} = \{X \in \text{Mod-}R \mid X \otimes_{\Lambda} \Lambda_{\mathcal{U}} = 0\}$.
(2) *Every* $A \in \text{Mod-}\Lambda$ *admits a short exact sequence*

$$0 \to A/tA \to A \otimes_{\Lambda} \Lambda_{\mathcal{U}} \to A \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda \to 0$$

where $A \otimes_{\Lambda} \Lambda_{\mathcal{U}} \in \mathcal{U}^{\wedge}$ *and* $A \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda \in \mathcal{T}$.

Proposition 4.4. *Let* $\mathcal{U}$ *be a set of simple regular modules.*
(1) $\Lambda_{\mathcal{U}}$ *is a torsion-free, and* $\Lambda_{\mathcal{U}}/\Lambda$ *is a torsion regular* Λ*-module. If* $\mathcal{U}$ *is a union of cliques, then* $\Lambda_{\mathcal{U}}/\Lambda$ *is a direct sum of all Prüfer modules from the corresponding tubes.*
(2) *If* $\mathcal{U}$ *does not contain a complete clique, then* $\Lambda_{\mathcal{U}}$ *is a concealed canonical algebra with canonical trisection* $(\mathbf{p}_{\mathcal{U}}, \mathbf{t}_{\mathcal{U}}, \mathbf{q}_{\mathcal{U}})$*, and the functors* λ_* *and* λ^* *map* $\operatorname{add} \mathbf{p}_{\mathcal{U}}$ *to* $\operatorname{add} \mathbf{p}$*,* $\operatorname{add} \mathbf{t}_{\mathcal{U}}$ *to* $\operatorname{add} \mathbf{t}$*,* $\operatorname{add} \mathbf{q}_{\mathcal{U}}$ *to* $\operatorname{add} \mathbf{q}$*, and viceversa. In particular,*
 (a) *the simple regular* $\Lambda_{\mathcal{U}}$*-modules are precisely the modules of the form* $S \otimes_{\Lambda} \Lambda_{\mathcal{U}}$ *where* $S \notin \mathcal{U}$ *is simple regular;*
 (b) *the Prüfer modules over* $\Lambda_{\mathcal{U}}$ *are precisely the modules of the form* $S_{\infty} \otimes_{\Lambda} \Lambda_{\mathcal{U}} \cong S_{\infty}$ *where* $S \notin \mathcal{U}$ *is simple regular;*
 (c) *every* $A \in \mathcal{U}^o$ *admits a short exact sequence* $0 \to A \to A \otimes_{\Lambda} \Lambda_{\mathcal{U}} \to A \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda \to 0$*, where* $A \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda$ *has a finite filtration by modules in* $\operatorname{Add} \mathcal{U}$*, and thus lies in* ${}^{\perp}(\mathcal{U}^{\perp})$*;*
 (d) $\mathbf{L} \otimes_{\Lambda} \Lambda_{\mathcal{U}}$ *is the Lukas tilting module over* $\Lambda_{\mathcal{U}}$.

Proof. The first part of (2) is shown as in [36, Proposition 4.2 (Going down)], while (1) and (2)(b) are proven as in [6, Propositions 1.8, 1.10, and 1.11].
In order to prove 2(c), we assume w.l.o.g. that $\mathcal{U}$ consists of $m < r$ simple regular modules from a tube of rank r, and we proceed by induction on m.
For $m = 1$ we have $\mathcal{U} = \{S\}$ for a simple regular in a tube of rank $r > 1$, and λ_* is the embedding of the perpendicular category $S^{\wedge}$ of S. By the construction of the left adjoint λ^* in [19, 1.3] we know that the short exact sequence in Lemma 4.3 has the form $0 \to A \to A_0 \to S^{(c)} \to 0$ where c is the minimal number of generators of $\operatorname{Ext}^1_{\Lambda}(S, A)$ as a module over $\operatorname{End}_{\Lambda} S$.
Let now $1 < m < r$, and choose a numbering $\mathcal{U} = \{S_1, \ldots, S_m\}$ such that $\operatorname{Ext}^1_{\Lambda}(S_i, S_m) = 0$ for all $1 \leq i < m$. Then taking $\mathcal{U}' = \{S_1, \ldots, S_{m-1}\}$, we have that $S_m \in (\mathcal{U}')^{\wedge}$ is a regular $\Lambda_{\mathcal{U}'}$-module. So $\Lambda_{\mathcal{U}} \cong (\Lambda_{\mathcal{U}'})_{S_m}$ by [42, 4.6], and we can compute $A \otimes_{\Lambda} \Lambda_{\mathcal{U}} \cong (A \otimes_{\Lambda} \Lambda_{\mathcal{U}'}) \otimes_{\Lambda_{\mathcal{U}'}} (\Lambda_{\mathcal{U}'})_{S_m}$.
By induction assumption we have a short exact sequence

$$0 \longrightarrow A \longrightarrow A \otimes_{\Lambda} \Lambda_{\mathcal{U}'} \longrightarrow A \otimes_{\Lambda} \Lambda_{\mathcal{U}'}/\Lambda \longrightarrow 0$$

where $A \otimes_\Lambda \Lambda_{\mathcal{U}'}/\Lambda$ is a finite extension of modules in $\operatorname{Add}\mathcal{U}'$, and in particular, it belongs to $\{S_m\}^o$. Then also $A \otimes_\Lambda \Lambda_{\mathcal{U}'} \in \{S_m\}^o$. Arguing as in the case $m = 1$, we get an exact sequence $0 \to A \otimes_\Lambda \Lambda_{\mathcal{U}'} \to A \otimes_\Lambda \Lambda_{\mathcal{U}} \to S_m{}^{(c)} \to 0$ together with a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "="] & A \otimes_\Lambda \Lambda_{\mathcal{U}'} \arrow[r] \arrow[d] & A \otimes_\Lambda \Lambda_{\mathcal{U}'}/\Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & A \otimes_\Lambda \Lambda_{\mathcal{U}} \arrow[r] \arrow[d] & A \otimes_\Lambda \Lambda_{\mathcal{U}}/\Lambda \arrow[r] \arrow[d] & 0 \\
 & & S_m{}^{(c)} \arrow[r, "="] \arrow[d] & S_m{}^{(c)} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

which yields the claim.

In 2(d), we specialize to $A = \mathbf{L}$, which certainly belongs to $\mathcal{U}^o$ as it is torsion-free. We get a short exact sequence

$$0 \to \mathbf{L} \to \mathbf{L} \otimes_\Lambda \Lambda_{\mathcal{U}} \to \mathbf{L} \otimes_\Lambda \Lambda_{\mathcal{U}}/\Lambda \to 0$$

where the two outer terms have projective dimension at most one over Λ, hence so does $\mathbf{L} \otimes_\Lambda \Lambda_{\mathcal{U}}$. Since λ is a homological epimorphism, it follows that $\mathbf{L} \otimes_\Lambda \Lambda_{\mathcal{U}}$ is a $\Lambda_{\mathcal{U}}$-module of projective dimension at most one. The remaining part of the proof works as in [2, Theorem 6]. $\square$

5. Gabriel localizations of the heart

Aim of this section is to investigate the Gabriel localizations of $\mathcal{A} = \operatorname{Qcoh}\mathbb{X}$. This leads to classification results for tilting or cotilting modules over Λ. More precisely, we are going to classify the tilting and cotilting modules in the class

$$\mathcal{M} = \mathcal{B} \cap \mathcal{C}.$$

Observe that $\operatorname{add}\mathbf{t}$ is the class of finite dimensional modules in $\mathcal{M}$, so there are no finite dimensional tilting modules in $\mathcal{M}$. Indeed, given a tilting module T, the number of pairwise non-isomorphic indecomposable summands from $\mathbf{t}$ is bounded by $\sum_{i=1}^{t}(p_i - 1)$, where $p_1, \ldots, p_t$ are the ranks of the non-homogeneous tubes in $\mathbf{t}$, and it is therefore strictly smaller than the rank of the Grothendieck group $\operatorname{rk} K_0(\Lambda) = \sum_{i=1}^{t}(p_i - 1) + 2$.

On the other hand, the modules in $\operatorname{add}\mathbf{t}$ can occur as direct summands of a tilting module. Here is a first structure result.

Proposition 5.1. *Let T be a tilting module in $\mathcal{C}$. Every module $X \in \operatorname{Add}T$ has a unique decomposition $X = X' \oplus \overline{X}$ where $\overline{X}$ is torsion-free and X' is a direct sum of Prüfer modules and modules from $\mathbf{t}$.*

Proof. The proof of [6, Proposition 4.2], is still valid in our context. We only have to explain why the torsion part X' of a module $X \in \operatorname{Add}T$ is again in ${}^\perp(T^\perp)$. This can be seen by applying the functor $\operatorname{Hom}_\Lambda(-, B)$ with $B \in T^\perp$ on the canonical sequence $0 \to X' \to X \to \overline{X} \to 0$, keeping in mind that $\overline{X}$ is in $\mathcal{C}$ and thus has projective dimension at most one by Lemma 3.1. $\square$

One shows as in [6, §3] that the direct summands from $\mathbf{t}$ in a tilting module $T \in \mathcal{M}$ are arranged in disjoint wings, and the direct sum Y of a complete irredundant set of such summands is a module of the following form.

Definition A multiplicity free Λ-module $Y \in \operatorname{add}\mathbf{t}$ is called a *branch module* if it satisfies

(B1) $\operatorname{Ext}^1_\Lambda(Y, Y) = 0$,

(B2) for each simple regular module S and $m \in \mathbb{N}$ such that S_m is a direct summand of Y, there exist precisely m direct summands of Y that belong to $\mathcal{W}_{S_m}$.

Finite dimensional torsion summands can be "removed" by employing universal localization.

Proposition 5.2. *Let $T = Y \oplus M$ be a tilting module where $0 \neq Y \in \operatorname{add} \mathbf{t}$, and let $\mathcal{U}$ be the set of regular composition factors of Y. Assume that $M \in \mathcal{U}^o$. Then M is a tilting module over $\Lambda_{\mathcal{U}}$, which is large if and only if T is large. In particular, if M is a torsion-free Λ-module, then it is a torsion-free tilting module over $\Lambda_{\mathcal{U}}$.*

Proof. In order to show that M is a $\Lambda_{\mathcal{U}}$-module, we have to verify $M \in \mathcal{U}^{\perp}$. This is deduced inductively from the fact that $M \in Y^{\perp}$. In fact, if we assume w.l.o.g. that $Y = S_m$ is indecomposable, then $\operatorname{Ext}^1_{\Lambda}(S_i, M) = 0$ for all its regular submodules $S_i, i < m$, because Y/S_i has projective dimension one. Then, applying $\operatorname{Hom}(-, M)$ on the Auslander-Reiten sequences and keeping in mind that $\operatorname{Hom}(S_i, M) = 0$, one obtains $\operatorname{Ext}^1_{\Lambda}((\tau^- S)_i, M) = 0$ for all $i < m$.
Now, since $\lambda : \Lambda \to \Lambda_{\mathcal{U}}$ is a homological ring epimorphism, a $\Lambda_{\mathcal{U}}$-module N satisfies $\operatorname{Ext}^i_{\Lambda_{\mathcal{U}}}(M, N) = 0$ for some $i \geq 0$ if and only if $\operatorname{Ext}^i_{\Lambda}(T, N) = 0$. It follows immediately that M fulfills conditions (T1), (T2), and (T3) over $\Lambda_{\mathcal{U}}$.
As discussed above, $\mathcal{U}$ cannot contain a complete clique, so $\Lambda_{\mathcal{U}}$ is a concealed canonical algebra. From $\Lambda_{\mathcal{U}} \in \text{mod-}\Lambda$ we infer $\text{mod-}\Lambda_{\mathcal{U}} \subset \text{mod-}\Lambda$, which shows that M is equivalent to a finite dimensional tilting module over $\Lambda_{\mathcal{U}}$ if and only if so is T over Λ.
In the special case when M is torsion-free, the assumption $M \in \mathcal{U}^o$ is satisfied. Further, by Proposition 4.4, the simple regular $\Lambda_{\mathcal{U}}$-modules are precisely the modules of the form $S \otimes_{\Lambda} \Lambda_{\mathcal{U}}$ where $S \notin \mathcal{U}$ is simple regular, and for any such S there is an exact sequence $0 \to S \to S \otimes_{\Lambda} \Lambda_{\mathcal{U}} \to S \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda \to 0$, where $S \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda$ is a finite extension of modules in $\operatorname{Add} \mathcal{U}$. Applying $\operatorname{Hom}_{\Lambda}(-, M)$ we get $0 = \operatorname{Hom}_{\Lambda}(S \otimes_{\Lambda} \Lambda_{\mathcal{U}}/\Lambda, M) \to \operatorname{Hom}_{\Lambda}(S \otimes_{\Lambda} \Lambda_{\mathcal{U}}, M) \to \operatorname{Hom}_{\Lambda}(S, M) = 0$, showing that M is torsion-free over $\Lambda_{\mathcal{U}}$. □

5.1. Tilting modules in $\mathcal{M}$. Let now Y be a branch module, and let $\mathcal{U}$ be the set of all regular composition factors of Y. Denote

$$T_{(Y,\emptyset)} = Y \oplus (\mathbf{L} \otimes_{\Lambda} \Lambda_{\mathcal{U}}).$$

Moreover, given a non-empty subset $P \subset \mathbb{X}$, let $\mathcal{V}$ be the union of $\mathcal{U}$ with the cliques of the tubes $\mathcal{U}_x, x \in P$, and set

$$T_{(Y,P)} = Y \oplus \bigoplus \{\text{all } S_\infty \text{ in } {}^{\perp}Y \text{ from tubes } \mathcal{U}_x, x \in P\} \oplus \Lambda_{\mathcal{V}}$$

Dually, consider

$$C_{(Y,P)} = Y \oplus \prod \{\text{all } S_{-\infty} \text{ in } Y^{\perp} \text{ from } \mathcal{U}_x, x \in P\} \oplus G \oplus \bigoplus \{\text{all } S_\infty \text{ in } {}^{\perp}Y \text{ from } \mathcal{U}_x, x \notin P\}$$

Notice that by the Auslander-Reiten formula, $S_{-\infty} \in Y^{\perp}$ if and only if S does not occur as a regular composition factor of τY, and similarly, $S_\infty \in {}^{\perp}Y$ if and only if S does not occur as a regular composition factor of $\tau^- Y$.

We will prove that the $T_{(Y,P)}$ and the $C_{(Y,P)}$ give a complete list of all tilting, respectively cotilting, modules in $\mathcal{M}$, up to equivalence. We start by collecting some information on the class $\mathcal{M}$.

Theorem 5.3. [41, 2.2] *A Λ-module M is pure-injective and belongs to $\mathcal{M}$ if and only if there is a decomposition $M = M' \oplus M''$ where $M' \in \operatorname{Prod} \mathbf{t}$ and $M'' \in \operatorname{Add} \mathbf{W}$.*

Lemma 5.4. *Let T be a tilting module, and let $\mathcal{S} = {}^{\perp}(T^{\perp}) \cap \text{mod-}\Lambda$ be the resolving subcategory corresponding to T under the bijection of Theorem 2.1.*

(1) *$T \in \mathcal{C}$ if and only if $\mathcal{S} \subset \mathcal{C}$.*
(2) *$T \in \mathcal{B}$ if and only if $\mathbf{p} \subset \mathcal{S}$.*
(3) *$T \in \mathcal{M}$ if and only if $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ for some subset $\mathbf{t}' \subset \mathbf{t}$. In particular, $\mathcal{S}$ is then closed under submodules.*

(4) *Assume $T \in \mathcal{C}$. Then the ray of a simple regular module S is completely contained in $\mathcal{S}$ if and only if the corresponding Prüfer module S_∞ is a direct summand of T. On the other hand, if $\mathcal{S}$ contains some, but not all modules from that ray, then T has a direct summand $S_m \in \mathcal{S}$ (which is the module of maximal regular length in $\mathbf{t}' \cap \{S_n \mid n \in \mathbb{N}\}$).*

Proof. (1) First of all, recall that the cotilting class $\mathcal{C} = {}^\perp(\mathcal{C}^\perp)$ is closed under direct summands and filtrations, cf. [25, 3.1.2]. The if-part then follows from the fact that T is a direct summand of an $\mathcal{S}$-filtered module by [25, 3.2.4]. Conversely, $T \in \mathcal{C}$ implies $\mathcal{C}^\perp \subset T^\perp$, and ${}^\perp(T^\perp) \subset {}^\perp(\mathcal{C}^\perp) = \mathcal{C}$, hence $\mathcal{S} \subset \mathcal{C}$.
(2) Since $\mathcal{B}$ is a torsion class, $T \in \mathcal{B}$ if and only if $T^\perp = \operatorname{Gen} T \subset \mathcal{B} = \mathbf{p}^\perp$, that is, ${}^\perp(\mathbf{p}^\perp) \subset {}^\perp(T^\perp)$, and as above we see that the latter is equivalent to $\mathbf{p} \subset \mathcal{S}$.
(3) The first statement follows combining (1) and (2). Now, if $A' \in \operatorname{add} \mathbf{t}$ is a submodule of $A \in \operatorname{add} \mathbf{t}'$, then $A/A' \in \operatorname{add} \mathbf{t}$ has projective dimension one, hence $A' \in {}^\perp(T^\perp)$ if so does A. This shows that $\mathcal{S}$ is closed under submodules.
(4) is shown as in [6, 4.5 and 3.3]. □

Now we are ready for our classification result.

Theorem 5.5. *There are one-one-correspondences between*

(1) *the pairs (Y, P) where Y is a branch module and $P \subset \mathbb{X}$,*
(2) *the equivalence classes of tilting modules in $\mathcal{M}$,*
(3) *the equivalence classes of cotilting modules in $\mathcal{M}$,*

assigning to (Y, P) the (equivalence class of) the tilting module $T_{(Y,P)}$ and the cotilting module $C_{(Y,P)}$ defined above.

Proof. The proof of the bijection between (1) and (2) works as in [6, Theorem 5.6], we give an outline and point out the arguments that have to be modified.
Given a tilting module T in $\mathcal{M}$, let Y be the branch module obtained as direct sum of a complete irredundant set of finite dimensional indecomposable summands of T, and let $\mathcal{U}$ be the set of regular composition factors of Y.
According to Lemma 5.4, there is a subset $\mathbf{t}' \subset \mathbf{t}$ such that $\mathcal{S} = {}^\perp(T^\perp) \cap \text{mod-}\Lambda = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$. Moreover, the ray of a simple regular module S is completely contained in $\mathbf{t}'$ if and only if the corresponding Prüfer module S_∞ is a direct summand of T.

The first part of the proof now consists in showing that

(i) If $\mathbf{t}'$ contains no complete ray from a tube in $\mathbf{t}$, then T is equivalent to a tilting module of the form $T_{(Y,\emptyset)} = Y \oplus (\mathbf{L} \otimes \Lambda_\mathcal{U})$.
(ii) If $\mathbf{t}'$ contains some rays, then T is equivalent to a tilting module of the form $T_{(Y,P)}$ where $P = \{x \in \mathbb{X} \mid \mathbf{t}' \text{ has rays from } \mathcal{U}_x\}$.

More precisely, one shows that T is equivalent to a tilting module of the form $T' = Y \oplus M$, where M is a tilting module over the universal localization at a suitable set of simple regular modules $\mathcal{U}'$ that contains no complete clique, and further, M is chosen such that the corresponding resolving subcategory of mod-$\Lambda_{\mathcal{U}'}$ is the localization $\mathcal{S} \otimes \Lambda_{\mathcal{U}'} = \{A \otimes_\Lambda \Lambda_{\mathcal{U}'} \mid A \in \mathcal{S}\}$ of $\mathcal{S}$.
In case (i), $\mathbf{t}'$ is contained in the extension closure of $\mathcal{U}$ by Lemma 5.4. We take $\mathcal{U}' = \mathcal{U}$, hence $\mathcal{S} \otimes \Lambda_{\mathcal{U}'} = \operatorname{add} \mathbf{p}_\mathcal{U}$, and M is the Lukas module over $\Lambda_\mathcal{U}$, that is $M = \mathbf{L} \otimes_\Lambda \Lambda_\mathcal{U}$ by Proposition 4.4. So $T' = T_{(Y,\emptyset)}$.
In case (ii), $\mathbf{t}'$ consists of the rays it contains (from tubes $\mathcal{U}_x, x \in P$) and of a subset of the extension closure of $\mathcal{U}$. Take $\mathcal{U}'$ as follows: $\mathcal{U}' \cap \mathcal{U}_x = \mathcal{U} \cap \mathcal{U}_x$ if $x \notin P$, while if $x \in P$, then $\mathcal{U}' \cap \mathcal{U}_x$ consists of the simple regular modules whose ray is not completely contained in $\mathbf{t}'$. Further, let $\mathcal{V}$ be the union of $\mathcal{U}'$ (or equivalently, of $\mathcal{U}$) with the cliques of the tubes $\mathcal{U}_x, x \in P$. Then the localization $\mathbf{t}' \otimes \Lambda_{\mathcal{U}'}$ of $\mathbf{t}'$ at $\mathcal{U}'$ coincides with the localization of the tubes $\mathcal{U}_x, x \in P$, and it is given by the set of simple regular $\Lambda_{\mathcal{U}'}$-modules $\mathcal{V}' = \{S \otimes \Lambda_{\mathcal{U}'} \mid S \in \mathcal{V} \setminus \mathcal{U}'\}$ corresponding to the simple regular Λ-modules S whose ray is contained in $\mathbf{t}'$. This shows that $\mathcal{S} \otimes \Lambda_{\mathcal{U}'}$ is the additive closure of $\mathbf{p}_{\mathcal{U}'}$ and a union of tubes, and the corresponding tilting class

is $\{X \in \text{Mod-}\Lambda_{\mathcal{U}'} \mid \operatorname{Ext}^1_{\Lambda_{\mathcal{U}'}}(V', X) = 0 \text{ for all } V' \in \mathcal{V}'\}$. Applying Theorem 4.2 on the canonical algebra $\Lambda_{\mathcal{U}'}$, and keeping in mind that $(\Lambda_{\mathcal{U}'})_{\mathcal{V}'} \cong \Lambda_{\mathcal{V}}$ by [42, 4.6], we conclude that we can take $M = \Lambda_{\mathcal{V}} \oplus \Lambda_{\mathcal{V}}/\Lambda_{\mathcal{U}'}$. Furthermore one infers from Proposition 4.4 that $\Lambda_{\mathcal{V}}/\Lambda_{\mathcal{U}'} \cong \bigoplus\{S_\infty \mid S \in \mathcal{V}\backslash\mathcal{U}'\}$, which by definition of $\mathcal{V}$ and $\mathcal{U}'$ is isomorphic to $\bigoplus\{\text{all } S_\infty \text{ in } {}^{\perp}Y \text{ from tubes } \mathcal{U}_x, x \in P\}$. Hence $T' = T_{(Y,P)}$.

In order to prove that T is actually equivalent to the tilting module $T' = Y \oplus M$ with M as explained above, one proceeds as in the proof of [6, Propositions 5.4 and 5.5] by verifying

(i) if $\mathcal{W}$ is the extension closure of $\mathcal{U}'$, then $\mathcal{W} \cup \mathcal{W}^o$ contains $\mathbf{t}'$,
(ii) $\operatorname{Add}(\mathbf{t} \cap \operatorname{Gen} T) \subset M^{\perp}$,
(iii) every torsion-free module in $\operatorname{Prod} \mathbf{t} \cap \operatorname{Gen} T$ is contained in $\mathcal{U}'^{\perp}$.

These conditions are an adapted version of [6, Proposition 5.2], where (iii) has been modified in view of the classification of pure-injectives in Theorem 5.3. The proof of [6, Proposition 5.2] works also in this context. In fact, we only have to change the argument for checking that the module M is in $T^{\perp}$ on [6, page 31, lines 23-26]: in order to verify that every $A \in \mathcal{S} \cap \mathcal{U}^o$ belongs to ${}^{\perp}M$, we use the exact sequence $0 \to A \to A \otimes_\Lambda \Lambda_{\mathcal{U}} \to A \otimes_\Lambda \Lambda_{\mathcal{U}}/\Lambda \to 0$ from Proposition 4.4 where $A \otimes_\Lambda \Lambda_{\mathcal{U}}/\Lambda$ has projective dimension one. Then $A \in {}^{\perp}M$ whenever $A \otimes_\Lambda \Lambda_{\mathcal{U}} \in {}^{\perp}M$, and the latter holds true by choice of M. Notice that [6, Proposition 5.2] relies on [6, Lemma 5.1]. For proving the latter in our context, it remains to explain why condition (T1) is verified: this follows from the assumption $T^{\perp} \subset M^{\perp}$, which yields that M belongs to the class ${}^{\perp}(T^{\perp})$ consisting of modules of projective dimension at most one.

The second part of the proof is devoted to establishing the stated bijection. Given a pair (Y, P) as in the Theorem, one proceeds as in the proof of [6, Theorem 5.6] to construct a tilting module $T \in \mathcal{M}$ which, according to the first part of the proof, must be equivalent to $T_{(Y,P)}$. So the assignment $(Y, P) \mapsto T_{(Y,P)}$ is well defined and surjective. For the injectivity, one uses Proposition 5.1 to see that the equivalence class of $T_{(Y,P)}$ determines the torsion part of the tilting module and thus the pair (Y, P).

The last part of the proof is devoted to the bijection between (1) and (3). First of all, since the dual of a branch right module Y is a branch left module $Y' = D(Y)$, we also have a bijection between the pairs in (1) and the equivalence classes of left tilting Λ-modules in the corresponding class ${}_\Lambda\mathcal{M}$ in Λ-Mod. But we know from Theorem 2.1 that the duality D yields a bijection between left tilting and right cotilting Λ-modules, and it is clear that $T \in {}_\Lambda\mathcal{M}$ if and only if $D(T)$ is in $\mathcal{M}$. So, we obtain also a bijection between (1) and (3).

It only remains to verify that the duals of the left tilting modules $T_{(Y',P)}$ are of the form $C_{(Y,P)}$ with $Y = D(Y')$, up to equivalence. Certainly, $C = D(T_{(Y',P)})$ is a cotilting right module isomorphic to

$$Y \oplus \prod\{\text{all } S_{-\infty} \text{ in } Y^{\perp} \text{ from tubes } \mathcal{U}_x, x \in P\} \oplus D(M)$$

where $M \in {}_\Lambda\mathcal{M}$ is the torsion-free part of $T_{(Y',P)}$. Then $D(M) \in \mathcal{M}$ is divisible and pure-injective, and by Theorem 5.3 it is a direct sum of Prüfer modules and copies of G. Of course, the Prüfer modules occurring in $\operatorname{Prod} C$ must lie in ${}^{\perp}Y$. Moreover, they must belong to tubes $\mathcal{U}_x, x \notin P$, because one can show as in [14, 2.7] that $\operatorname{Ext}^1_\Lambda(S_\infty, S'_{-\infty}) = 0$ if and only if and S and S' do not belong to the same clique.

Conversely, any such Prüfer module S_∞ is Ext-orthogonal to Y, to all adic modules $S_{-\infty}$ from tubes $\mathcal{U}_x, x \in P$, to G and to any Prüfer module, and therefore it lies in ${}^{\perp}C = \operatorname{Cogen} C$. We claim that it even belongs to $\operatorname{Prod} C$.

In fact, since $C \in \mathcal{C} = {}^{\perp}\mathbf{q}$, we have $\operatorname{Cogen} C \subset \mathcal{C}$ and therefore $\operatorname{Cogen} C \cap \text{mod-}\Lambda \subset \operatorname{add}(\mathbf{p} \cup \mathbf{t})$. As in the proof of [6, Theorem A.1], we infer that all pure-injective divisible modules, hence in particular G and the Prüfer modules, belong to $({}^{\perp}C)^{\perp}$. So the Prüfer modules $S_\infty \in \operatorname{Cogen} C$ belong to ${}^{\perp}C \cap ({}^{\perp}C)^{\perp} = \operatorname{Prod} C$. As for the generic module, recall that G is torsion-free and thus $G \in \varinjlim \mathbf{p}$. Then using that $C \in \mathcal{B} = \mathbf{p}^{\perp}$ and that ${}^{\perp}C$ is closed under direct limits, we deduce that $G \in {}^{\perp}C$ and therefore also $G \in \operatorname{Prod} C$.

So, we conclude that, up to multiplicities, $D(M) \cong G \oplus \bigoplus\{\text{all } S_\infty \text{ in } {}^\perp Y \text{ from tubes } \mathcal{U}_x, x \notin P\}$ and thus C is equivalent to $C_{(Y,P)}$. □

5.2. Gabriel localizations of $\mathcal{A}$. We now turn to the relationship with Gabriel localization. Recall that $\mathcal{A} = \operatorname{Qcoh}\mathbb{X}$ is a locally noetherian hereditary Grothendieck category, with $\operatorname{Inj}\mathcal{A} = \{G[1]\} \cup \{S_\infty[1] \mid S \text{ simple regular}\}$ being the set of indecomposable injective objects in $\mathcal{A}$. We consider the *Gabriel topology* on $\operatorname{Inj}\mathcal{A}$ with closed sets

$$I(\mathfrak{S}) = \{E \in \operatorname{Inj}\mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(C,E) = 0 \text{ for all } C \in \mathfrak{S}\}$$

where $\mathfrak{S}$ runs through the Serre subcategories of $\operatorname{fp}\mathcal{A}$. The torsion pair $(\mathcal{T},\mathcal{F})$ cogenerated by $I(\mathfrak{S})$ coincides with the one generated by $\mathfrak{S}$ and is a hereditary torsion pair with $\mathcal{T} = \varinjlim \mathfrak{S}$. The assignments $\mathfrak{S} \mapsto I(\mathfrak{S})$, and $\mathfrak{S} \mapsto (\mathcal{T},\mathcal{F})$ are part of the following correspondence.

Theorem 5.6. ([22],[29, 2.8 and 3.8],[31],[38, Ch. 11]) *There is a bijection between*

(1) *the hereditary torsion pairs in* $\mathcal{A}$*,*
(2) *the closed subsets of* $\operatorname{Inj}\mathcal{A}$*,*
(3) *and the Serre subcategories of* $\operatorname{fp}\mathcal{A}$*,*

which assigns to a hereditary torsion pair $(\mathcal{T},\mathcal{F})$ *the closed subset* $\mathcal{I} = \operatorname{Inj}\mathcal{A} \cap \mathcal{F}$ *and the Serre subcategory* $\mathfrak{S} = \operatorname{fp}\mathcal{A} \cap \mathcal{T}$*.*

If a Serre subcategory $\mathfrak{S} \subset \operatorname{fp}\mathcal{A}$ contains an indecomposable vector bundle, then it contains all vector bundles, and therefore the corresponding torsion theory $(\mathcal{T},\mathcal{F}) = (\mathcal{A},0)$ is trivial, cf. [23, 9.2]. So the hereditary torsion pairs $(\mathcal{T},\mathcal{F})$ in $\mathcal{A}$ with non-trivial $\mathcal{F}$ correspond to the Serre subcategories consisting of finite length objects. Then $\operatorname{vect}\mathbb{X} \subset \mathcal{F}$, or equivalently, $\mathcal{T} \subset {}^o(\operatorname{vect}\mathbb{X})$, and $(\mathcal{T},\mathcal{F})$ is faithful since $\operatorname{vect}\mathbb{X}$ contains a set of generators of $\mathcal{A}$.
Recall that the category of finite length objects in $\mathcal{A}$ is given by the tubular family $\mathcal{H}_0 = \mathbf{t}[1] = \bigcup_{x\in\mathbb{X}} \mathcal{U}_x[1]$. One easily verifies that the Serre subcategories of $\operatorname{add}\mathcal{H}_0$ are precisely the small additive closures of unions of tubes and wings in $\mathbf{t}[1]$.

Corollary 5.7. *There is a surjective map from the set of equivalence classes of cotilting modules in* $\mathcal{M}$ *to the set of faithful hereditary torsion pairs in* $\mathcal{A}$*. It assigns to the (equivalence class of the) cotilting module* $C_{(Y,P)}$ *the torsion pair* $(\mathcal{T},\mathcal{F})$ *in* $\mathcal{A}$ *cogenerated by the indecomposable injective summands of* $C_{(Y,P)}[1]$*.*

Proof. Let $I = \{G[1]\} \cup \{S_\infty[1] \mid S_\infty \text{ in } {}^\perp Y \text{ from } \mathcal{U}_x, x \notin P\}\} \subset \operatorname{Inj}\mathcal{A}$ be the set of indecomposable injective summands of $C_{(Y,P)}[1]$. It is easy to see that $I = I(\mathfrak{S})$ for the Serre subcategory $\mathfrak{S} = \mathfrak{S}_{(Y,P)} = \operatorname{add}(\bigcup_{x\in P} \mathcal{U}_x[1] \cup \bigcup_{i=1}^r \mathcal{W}_i[1])$ where $\mathcal{W}_1, \ldots, \mathcal{W}_r$ are the wings defined by the regular composition factors of $\tau^- Y$. Since all Serre subcategories of $\operatorname{add}\mathcal{H}_0$ have this form, the assignment $(Y,P) \mapsto \mathfrak{S}_{(Y,P)}$ is surjective. Now Theorems 5.5 and 5.6 yield the statement. □

Clearly this map is not injective in general, because different branch modules can give rise to the same wings. Moreover, we point out that $(\mathcal{T},\mathcal{F})$ need not coincide with the torsion pair given by the torsion-free class $\operatorname{Cogen} C_{(Y,P)}[1]$. For example, if $\mathcal{U}_x$ is a tube of rank 3, S is a simple regular in $\mathcal{U}_x$ and $S' = \tau^- S$, then $Y = S_2 \oplus S'$ is a branch module and $C = C_{(Y,\emptyset)} = G \oplus Y \oplus S_\infty \oplus \bigoplus$(all Prüfer modules from the other tubes) is a cotilting module that gives rise to a non-hereditary torsion pair in $\mathcal{A}$. In fact, $\operatorname{Cogen} C[1]$ is not closed under injective envelopes since it does not contain the injective envelope $S'_\infty[1]$ of $S'[1]$.
In the homogeneous case, however, the parametrization in Theorem 5.5 reduces to the subsets $P \subset \mathbb{X}$. So, denoting by Λ_P the universal localization at the cliques of the tubes $\mathcal{U}_x, x \in P$, we have tilting modules $T_P = \Lambda_P \oplus \Lambda_P/\Lambda$ when $P \neq \emptyset$, and $T_\emptyset = \mathbf{L}$, as well as cotilting modules $C_P = G \oplus \prod\{\text{all } S_{-\infty} \text{ from } \mathcal{U}_x, x \in P\} \oplus \bigoplus\{\text{all } S_\infty \text{ from } \mathcal{U}_x, x \notin P\}$. We then obtain a similar classification result as for commutative noetherian rings, cf. [4].

Corollary 5.8. *Assume that* Λ *is a tame bimodule algebra. There is a bijection between*

(1) *the subsets of* $\mathbb{X}$,
(2) *the equivalence classes of tilting modules in* $\mathcal{M}$,
(3) *the equivalence classes of cotilting modules in* $\mathcal{M}$,
(4) *the faithful hereditary torsion pairs in* $\mathcal{A}$.

The bijection assigns to a subset $P \subset \mathbb{X}$ *the tilting module* T_P, *the cotilting module* C_P, *and the faithful hereditary torsion pair* $(\mathcal{T}_P, \mathcal{F}_P)$ *in* $\mathcal{A}$ *given by* $\mathcal{T}_P = \varinjlim(\bigcup_{x\in P}\mathcal{U}_x[1])$ *and* $\mathcal{F}_P = \operatorname{Cogen} C_P[1]$. *When* $P \neq \emptyset$, *the quotient category* $\mathcal{A}/\mathcal{T}_P$ *is equivalent to* Mod-Λ_P.

Proof. The bijection between the sets in (1) - (4) follows from Theorem 5.5 and the discussion above. We only have to verify that the torsion pair $(\mathcal{T}_P, \mathcal{F}_P)$ in $\mathcal{A}$ cogenerated by the indecomposable injective summands of $C_P[1]$ has the stated shape. We have already seen in the proof of Corollary 5.7 that it coincides with the torsion pair generated by the Serre subcategory $\mathfrak{S}_P = \operatorname{add}(\bigcup_{x\in P}\mathcal{U}_x[1])$, so $\mathcal{T}_P$ looks as desired. Clearly, $\mathcal{F}_P \subset \operatorname{Cogen} C_P[1]$. The reverse inclusion follows from the fact that $G[1]$ cogenerates all torsion-free objects in $\mathcal{A}$ by [39, 4.1]. Further, it is well known (see e.g.[31]) that

$$\mathcal{A}/\mathcal{T}_P = \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(Y,X) = \operatorname{Ext}^1_{\mathcal{A}}(Y,X) = 0 \text{ for all } Y \in \mathfrak{S}_P\},$$

and by Proposition 4.2

$$\text{Mod-}\Lambda_P = \{X \in \text{Mod-}\Lambda \mid \operatorname{Hom}_{\Lambda}(Y,X) = \operatorname{Ext}^1_{\Lambda}(Y,X) = 0 \text{ for all } Y \in \bigcup_{x\in P}\mathcal{U}_x\}.$$

By assumption, $\mathcal{Q} = \operatorname{Add}\mathbf{q}$ and no module from $\mathbf{q}$ can belong to $(\bigcup_{x\in P}\mathcal{U}_x)^o$, hence Mod-$\Lambda_P \subset \mathcal{C}$. Similarly, $\mathcal{A}/\mathcal{T}_P \subset \mathcal{C}[1]$. So, the functor H_V from Section 2.5 yields the desired equivalence. □

We will see below that in the situation of Corollary 5.8 a (co)tilting module belongs to $\mathcal{M}$ if and only if it is large.

6. Large tilting and cotilting modules

This Section aims at a classification of the large tilting and cotilting modules for concealed canonical algebras of domestic or tubular representation type.

We start out by recalling that cotilting modules over noetherian rings are determined up to equivalence by their indecomposable summands, which are known to be pure-injective by [8].

Theorem 6.1. [47] *Let* C *be a cotilting left* R*-module over a left noetherian ring* R. *Then* $\operatorname{Prod} C$ *contains a family of indecomposable modules* $(M_i)_{i\in I}$ *such that* C *is a direct summand in a direct limit of modules in* $\operatorname{Prod}\{M_i \mid i \in I\}$ *and* ${}^{\perp}C = \bigcap_{i\in I} {}^{\perp}M_i$.

Proof. see [47, Theorem 3.7] and its proof. □

In the domestic case, the classification of large tilting modules has already been accomplished.

Theorem 6.2. *There are one-one-correspondences between*

(i) *the pairs* (Y, P) *where* Y *is a branch module and* $P \subset \mathbb{X}$,
(ii) *the equivalence classes of large tilting* Λ*-modules,*
(iii) *the equivalence classes of large cotilting* Λ*-modules.*

Proof. The statement will follow from Theorem 5.5 once we prove that all large tilting or cotilting modules are in $\mathcal{M}$. Let T be a large tilting right module, and let $C = D(T)$ be the dual cotilting left module. As in [6, 2.6], we infer that C has to be large as well.

Take a family of pairwise non-isomorphic indecomposable modules $(M_i)_{i\in I}$ in $\operatorname{Prod} C$ as in Theorem 6.1. The M_i cannot be all finite dimensional. Indeed, otherwise the cardinality of the index set I is bounded by the rank of the Grothendieck group, and so the module $M = \prod_{i\in I} M_i$ is finite dimensional, and $\operatorname{Add} M = \operatorname{Prod} M$ is definable. Since C is a direct summand of a direct limit of modules in $\operatorname{Prod} M$, we infer that $C \in \operatorname{Add} M$ is equivalent to a finite dimensional cotilting module, a contradiction.

So there is $i \in I$ such that M_i is infinite dimensional, and by Proposition 3.5 it is isomorphic to a Prüfer or adic module, or to the generic module. Now Lemma 3.3 shows that no M_i can belong to $\mathcal{Q}$ nor to $\mathcal{P}$, hence all M_i are in $\mathbf{t}$, or Prüfer, adic, or generic, and in particular they all belong to $\mathcal{M}$. But $\mathcal{M}$ is definable, so also C, as a direct summand of a direct limit of modules in $\operatorname{Prod} M$, belongs to $\mathcal{M}$. And by duality T has to belong to $\mathcal{M}$ as well. □

From now on, we assume that Λ is concealed canonical of tubular type. The AR-quiver of Λ then consists of a preprojective component $\mathbf{p}_0$, a preinjective component $\mathbf{q}_\infty$ and a countable number of sincere separating tubular families $\mathbf{t}_\alpha, \alpha \in \mathbb{Q}_0^\infty = \mathbb{Q}^+ \cup \{0, \infty\}$, where $\mathbf{t}_\alpha$ is stable precisely when $\alpha \in \mathbb{Q}^+$. We fix $w \in \mathbb{R}^+$ and set

$$\mathbf{p}_w = \mathbf{p}_0 \cup \bigcup_{\alpha < w} \mathbf{t}_\alpha, \qquad \mathbf{q}_w = \bigcup_{w < \gamma} \mathbf{t}_\gamma \cup \mathbf{q}_\infty$$

For $w \in \mathbb{Q}^+$ we thus obtain a trisection $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$ as in Section 3.1, while for $w \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ the finitely generated indecomposable modules all belong either to $\mathbf{p}_w$ or to $\mathbf{q}_w$.

6.1. The slope of a module. Following [39, §13], we now consider torsion pairs in Mod-Λ constructed from the classes above and use them to define a notion of slope. We start with the torsion pair cogenerated by $\mathbf{p}_w$. By the Auslander-Reiten formula

$$\mathcal{B}_w = {}^o(\mathbf{p}_w) = (\mathbf{p}_w)^\perp$$

with $\operatorname{add} \mathbf{p}_w$ being resolving, so it follows from Theorem 2.1 that $\mathcal{B}_w$ is a tilting class. The corresponding torsion–free class will be denoted by $\mathcal{P}_w$.
We fix a tilting module $\mathbf{L}_w$ generating $\mathcal{B}_w$. Notice that it must be infinite dimensional: otherwise $\mathbf{L}_w \in \mathcal{B}_w \cap {}^\perp(\mathcal{B}_w) \cap \text{mod-}\Lambda = \mathcal{B}_w \cap \operatorname{add} \mathbf{p}_w = {}^o(\mathbf{p}_w) \cap \operatorname{add} \mathbf{p}_w$, which is impossible.
Furthermore, $\mathcal{B}_w = {}^o(\bigcup_{\alpha<w} \mathbf{t}_\alpha)$. Indeed, if a module X has a non-zero morphism $X \to P$ for some $P \in \mathbf{p}_0$, then we consider the injective envelope $f : P \to I(P)$. Since all indecomposable injective modules lie in $\mathbf{t}_\infty \cup \mathbf{q}_\infty$, we can factor through any tube in any tubular family $\mathbf{t}_\alpha$ where $\alpha < w$. This shows that P embeds in a module in $\operatorname{add} \mathbf{t}_\alpha$, and so we obtain a non-zero morphism $X \to Y$ with $Y \in \mathbf{t}_\alpha$.
We now turn to the torsion pair generated by $\mathbf{q}_w$. By the Auslander-Reiten formula

$$\mathcal{C}_w = (\mathbf{q}_w)^o = {}^\perp(\mathbf{q}_w).$$

Since $\mathbf{q}_w$ is dual to a class ${}_\Lambda\mathbf{p}_{\tilde{w}} \subset \Lambda$-mod, by the well-known Ext-Tor formulae and Theorem 2.1 it follows from that $\mathcal{C}_w = ({}_\Lambda\mathbf{p}_{\tilde{w}})^\intercal$ is a cotilting class given by a large cotilting module $\mathbf{W}_w$. Dually, $\mathcal{C}_w = (\bigcup_{w<\gamma} \mathbf{t}_\gamma)^o$. The corresponding torsion class is $\mathcal{Q}_w = \operatorname{Gen} \mathbf{q}_w$, and for $w \in \mathbb{Q} \cup \{0, \infty\}$ the torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ is split, see [39, 13.1].
Finally, let $\mathbf{p}_\infty = \mathbf{p}_0 \cup \bigcup_{\alpha<\infty} \mathbf{t}_\alpha$ and $\mathbf{q}_0 = \bigcup_{0<\gamma} \mathbf{t}_\gamma \cup \mathbf{q}_\infty$, and set as above $\mathcal{B}_w = {}^o(\mathbf{p}_w)$, and $\mathcal{C}_w = (\mathbf{q}_w)^o$ for $w = 0$ or $w = \infty$. According to [39], the modules belonging to the class

$$\mathcal{M}_w = \mathcal{C}_w \cap \mathcal{B}_w$$

with $w \in \mathbb{R}^+ \cup \{0, \infty\}$ will be said to have *slope* w.

6.2. Rational slope. When $w \in \mathbb{Q}^+$, we are in the situation of Sections 3.1 and 3.3. So there is a tilting and cotilting module $\mathbf{W}_w$ which cogenerates $\mathcal{C}_w$ and generates the torsion class $\mathcal{D}_w = {}^o(\mathbf{t}_w) = (\mathbf{t}_w)^\perp$ with corresponding split torsion pair $(\mathcal{D}_w, \mathcal{R}_w)$. The module $\mathbf{W}_w$ can be chosen as the direct sum of a set of representatives of the Prüfer modules and the generic module G_w from the family $\mathbf{t}_w$.
Furthermore, $\mathcal{D}_w = {}^o(\bigcup_{\alpha \le w} \mathbf{t}_\alpha) = {}^o(\mathbf{p}_w \cup \mathbf{t}_w)$. Indeed, if a module X has a non-zero morphism $X \to P$ for some $P \in \mathbf{p}_w$, then as shown above we can assume that $P \in \mathbf{t}_\alpha$ with $\alpha < w$. But since the modules in $\mathbf{t}_\alpha$ are cogenerated by $\mathbf{t}_w$, we infer that there is a non-zero morphism $X \to Y$ with $Y \in \mathbf{t}_w$.
Similarly, the torsion pair $(\operatorname{Gen} \mathbf{t}_w, \mathcal{F}_w)$ generated by $\mathbf{t}_w$ satisfies $\mathcal{F}_w = (\bigcup_{w \le \gamma} \mathbf{t}_\gamma)^o = (\mathbf{t}_w \cup \mathbf{q}_w)^o$.

We can now consider the t-structure induced by the torsion pair $(\operatorname{Gen}\mathbf{q}_w, \mathcal{C}_w)$ in $\mathcal{D}(\text{Mod-}\Lambda)$. Its heart by $\mathcal{A}_w$ is equivalent to the category $\operatorname{Qcoh}\mathbb{X}_w$ of quasi-coherent sheaves over a noncommutative curve of genus zero $\mathbb{X}_w$ parametrizing the family $\mathbf{t}_w$, which is again of tubular type by [36],[31, 8.1.6].
According to Theorem 5.5, the tilting and cotilting modules of slope w are then parametrized by the pairs (Y, P) where Y is a branch object in $\mathbf{t}_w$ and $P \subset \mathbb{X}_w$, and they are related to the Gabriel localizations of $\operatorname{Qcoh}\mathbb{X}_w$ as explained in Corollary 5.7.

6.3. Irrational slope. Let us first collect some properties of the classes introduced above.

Lemma 6.3. *Let $w \in \mathbb{R}^+$. Then*

(1) $\mathcal{B}_w = \bigcap_{\mathbb{R}^+ \ni v < w} \mathcal{B}_v = \bigcap_{\mathbb{Q}^+ \ni \alpha < w} \mathcal{D}_\alpha,$ *and* $\mathcal{C}_w = \bigcap_{w < v \in \mathbb{R}^+} \mathcal{C}_v = \bigcap_{w < \gamma \in \mathbb{Q}^+} \mathcal{F}_\gamma.$

(2) $\mathcal{Q}_w = \varinjlim \mathbf{q}_w$, *and if $w \notin \mathbb{Q}$, then* $\mathcal{C}_w = \varinjlim \mathbf{p}_w$.

(3) $\mathcal{P}_w \subset \mathcal{C}_w$ *and* $\mathcal{Q}_w \subset \mathcal{B}_w$. *If $w \in \mathbb{Q}^+$, then* $\mathcal{P}_w \subset \mathcal{F}_w \subset \mathcal{C}_w$ *and* $\mathcal{Q}_w \subset \mathcal{D}_w \subset \mathcal{B}_w$.

(4) $(\mathcal{C}_w)^\perp \subset \mathcal{B}_w = \bigcap_{\mathbb{R}^+ \ni v < w} (\mathcal{C}_v)^\perp = \bigcap_{\mathbb{R}^+ \ni v < w} \mathcal{Q}_v.$

Proof. (1) By definition, a module belongs to $\mathcal{B}_w$ if and only if it belongs to ${}^o(\mathbf{p}_0 \cup \bigcup_{\alpha < v} \mathbf{t}_\alpha) = \mathcal{B}_v$ for all $v < w$. Moreover, by the description of $\mathcal{B}_w$ in Section 6.1, we have $\mathcal{B}_w = \bigcap_{\alpha < w} {}^o(\mathbf{t}_\alpha) = \bigcap_{\alpha < w} \mathcal{D}_\alpha$, cf. also [39, 13.4]. The second statement is proven with dual arguments.
(2) The first statement follows from [20] or [25, 4.5.2] using that $\operatorname{add}\mathbf{q}_w$ is a torsion class in mod-Λ. For the second statement recall that every module is a direct limit of finitely presented modules, so by definition of $\mathcal{C}_w$ we obtain $\subset$. For the reverse inclusion, observe that $\mathbf{p}_w \subset \mathbf{q}_w^o = \mathcal{C}_w$ and $\mathcal{C}_w$ is closed under direct limits.
(3) follows immediately from the definitions and the separating condition.
(4) For $\alpha \in \mathbb{Q}^+$ we know e.g. from [39, §14] that $\mathcal{D}_\alpha = (\mathcal{C}_\alpha)^\perp$, and we infer that $\bigcap_{\mathbb{R}^+ \ni v < w} (\mathcal{C}_v)^\perp \subset \bigcap_{\mathbb{Q}^+ \ni \alpha < w} \mathcal{D}_\alpha = \mathcal{B}_w$. For the reverse inclusion, pick $B \in \mathcal{B}_w$ and $C \in \mathcal{C}_v$ where $v \in \mathbb{R}^+$ with $v < w$. Choose $\alpha \in \mathbb{Q}^+$ with $v < \alpha < w$. Then $\operatorname{Ext}^1_\Lambda(C, B) = 0$ because $C \in \mathcal{C}_\alpha$ and $B \in \mathcal{D}_\alpha$. For the second equality we refer to [39, 13.4]. Finally, since $\mathcal{C}_v \subset \mathcal{C}_w$ for all $v < w$, we have $(\mathcal{C}_w)^\perp \subset \mathcal{B}_w$. □

We obtain the following description of modules of irrational slope.

Theorem 6.4. *Let $w \in \mathbb{R}^+ \setminus \mathbb{Q}^+$.*

(1) $\mathbf{L}_w$ *is the only tilting module of slope w up to equivalence.*
(2) $\mathbf{W}_w$ *is the only cotilting module of slope w up to equivalence.*
(3) *A module has slope w if and only if it is a pure submodule of a product of copies of $\mathbf{W}_w$, or equivalently, it is a pure-epimorphic image of a direct sum of copies of $\mathbf{L}_w$.*

Proof. By construction, $\mathbf{L}_w \in \mathcal{B}_w \cap {}^\perp(\mathcal{B}_w) \subset \mathcal{B}_w \cap \mathcal{C}_w$, and $\mathbf{W}_w \in \mathcal{C}_w \cap (\mathcal{C}_w)^\perp \subset \mathcal{C}_w \cap \mathcal{B}_w$ have slope w.
(1) Let T be a tilting module of slope w. As in Lemma 5.4, it follows that the corresponding resolving subcategory $\mathcal{S} = {}^\perp(T^\perp) \cap \text{mod-}\Lambda = \operatorname{add}\mathbf{p}_w$, and Theorem 2.1 implies that T is equivalent to $\mathbf{L}_w$.
(2) Let now C be a cotilting module of slope w. Then ${}^\perp\mathcal{B}_w \subset {}^\perp C = \operatorname{Cogen} C \subset \mathcal{C}_w$. Since $\mathbf{p}_w \subset {}^\perp\mathcal{B}_w$ and ${}^\perp C$ is closed under direct limits, we infer $\varinjlim \mathbf{p}_w \subset {}^\perp C$. Lemma 6.3(2) now yields $\mathcal{C}_w = {}^\perp C$, so C is equivalent to $\mathbf{W}_w$.
(3) First of all, note that $\mathcal{M}_w$ is definable and therefore closed under direct sums, direct products, pure submodules and pure-epimorphic images, see for instance [9, 4.2 and 4.3]. So, we only have to prove the only-if part. Let M be a module of slope w. As shown in [7], there is a short exact sequence

$$0 \to M \to C_0 \to C_1 \to 0$$

where $C_0 \in (\mathcal{C}_w)^\perp$ and $C_1 \in \mathcal{C}_w$. As $M \in \mathcal{C}_w$, also the middle term $C_0 \in \mathcal{C}_w$. As $\mathcal{C}_w$ consists of modules of projective dimension at most one by [39, 5.4], the class $(\mathcal{C}_w)^\perp$ is closed under epimorphic images and thus $C_1 \in (\mathcal{C}_w)^\perp$. We conclude that $C_0, C_1 \in \mathcal{C}_w \cap (\mathcal{C}_w)^\perp = \operatorname{Prod} \mathbf{W}_w$.
It remains to prove that the sequence is pure-exact, that is, that it stays exact under the functor $\operatorname{Hom}_\Lambda(F, -)$ for any finitely generated Λ-module F. We can assume w.l.o.g. that F is indecomposable with $\operatorname{Hom}_\Lambda(F, C_1) \neq 0$, hence $F \in \mathbf{p}_w$. Then the claim follows from the fact that $\operatorname{Ext}^1_\Lambda(F, M) = 0$ as $M \in \mathcal{B}_w = \mathbf{p}_w{}^\perp$.
The second statement is proven dually. □

The tilting module $\mathbf{L}_w$ can be constructed in a similar way as the Lukas tilting module in [30, 37].

Proposition 6.5. *Let $w \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. Given a sequence of rational numbers $\alpha_1 < \alpha_2 < \ldots < w$ converging to w, there is a chain of modules in* $\operatorname{add} \mathbf{p}_w$

$$\Lambda = P_0 \subset P_1 \subset P_2 \subset \ldots$$

with P_i of slope α_i for all $i \geq 1$ and $P_{i+1}/P_i \in \operatorname{add} \mathbf{p}_w$ for all $i \geq 0$, such that the modules $L_0 = \bigcup_{i \in \mathbb{N}} P_i$ and $L_1 = L_0/\Lambda$ give rise to a tilting module $\mathbf{L}_w = L_0 \oplus L_1$ with tilting class $\operatorname{Gen} \mathbf{L}_w = \mathcal{B}_w$.

Proof. Start with $P_0 = \Lambda$. Given P_i, which by assumption belongs to $\operatorname{add} \mathbf{p}_{\alpha_{i+1}} \subset \mathcal{C}_w$, take a special $(\mathcal{C}_w)^\perp$-approximation as in [7]

$$0 \to P_i \xrightarrow{f} W_0 \to W_1 \to 0$$

with $W_0, W_1 \in \operatorname{Prod} W_w$. Observe that $W_0 \in (\mathcal{C}_w)^\perp \subset \mathcal{Q}_{\alpha_{i+1}} = \varinjlim \mathbf{q}_{\alpha_{i+1}}$ by Lemma 6.3 (2) and (4). So f factors through a map $P_i \xrightarrow{f'} Q$ with $Q \in \operatorname{add} \mathbf{q}_{\alpha_{i+1}}$, and by the separation condition f' factors through a map $P_i \xrightarrow{f''} P_{i+1}$ with $P_{i+1} \in \operatorname{add} \mathbf{t}_{\alpha_{i+1}}$ of slope α_{i+1}. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_i \arrow[r, "f''"] \arrow[d, "="] & P_{i+1} \arrow[r] \arrow[d, "h"] & Z_{i+1} \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & P_i \arrow[r, "f"] & W_0 \arrow[r] & W_1 \arrow[r] & 0
\end{tikzcd}$$

where $\operatorname{Ker} g \cong \operatorname{Ker} h \subset P_{i+1} \in \mathcal{C}_w$ cannot have submodules in $\mathbf{q}_w$. Then Z_{i+1} cannot have submodules $U \in \mathbf{q}_w$, because $g \mid_U \colon U \to W_1 \in \mathcal{C}_w$ would have to be zero and thus $U \subset \operatorname{Ker} g$. So, we conclude that $P_{i+1}/P_i \cong Z_{i+1} \in \operatorname{add} \mathbf{p}_w$.
The $\mathbf{p}_w$-filtered modules $L_0 = \bigcup_{i \in \mathbb{N}} P_i$ and $L_1 = L_0/\Lambda$ then belong to ${}^\perp(\mathbf{p}_w{}^\perp) = {}^\perp\mathcal{B}_w$ by [25, 3.2.4]. Further, the direct limit L_0 of the chain $P_1 \subset P_2 \subset \ldots$ has slope w by [39, 13.4]. This implies that L_0 and L_1 belong to $\mathcal{B}_w$ and therefore to $\operatorname{Add} \mathbf{L}_w = \mathcal{B}_w \cap {}^\perp\mathcal{B}_w$. Now the claim follows easily. □

6.4. Pure-injective modules. By Theorem 5.3, the pure-injective modules of rational slope w are precisely the modules of the form $M = M' \oplus M''$ where $M' \in \operatorname{Prod} \mathbf{t}_w$ and $M'' \in \operatorname{Add} \mathbf{W}_w = \operatorname{Prod} \mathbf{W}_w$. By results of Harland and Prest [28], pure-injective modules of irrational slope can be superdecomposable when the ground field k is countable. Observe, however, that the superdecomposable part does not play a role when computing the cotilting class $\mathcal{C}_w$, as shown by Theorem 6.1. Moreover, the class of pure-injectives is described as follows.

Corollary 6.6. *If $w \in \mathbb{R}^+ \setminus \mathbb{Q}^+$, then $\mathbf{W}_w$ is a pure-injective, non-Σ-pure-injective module, and* $\operatorname{Prod} \mathbf{W}_w$ *is the class of all pure-injective modules of slope w.*

Proof. The cotilting module $\mathbf{W}_w$ is pure-injective by [8], and $\operatorname{Prod} \mathbf{W}_w$ is the class of all pure-injective module of slope w by Theorem 6.4(3).
Assume that $\mathbf{W}_w$ is Σ-pure-injective. Then so is every product of copies of $\mathbf{W}_w$ and any pure submodule of such product, yielding $\mathcal{M}_w = \operatorname{Prod} \mathbf{W}_w$ by Theorem 6.4(3). Hence $\mathbf{L}_w \in \mathcal{M}_w \subset$

$(\mathcal{C}_w)^\perp$, and as $\mathcal{C}_w$ consists of modules of projective dimension at most one, $\operatorname{Gen}\mathbf{L}_w \subset (\mathcal{C}_w)^\perp$, and $(\mathcal{C}_w)^\perp = \mathcal{B}_w$ by Lemma 6.3(4). From $\mathcal{Q}_w \subset \mathcal{B}_w$ we deduce that $(\mathcal{Q}_w, \mathcal{C}_w)$ is a split torsion pair satisfying the assumptions of Proposition 2.5 (for assumption (v) observe that $\mathcal{C}_w \subset \mathcal{C}_v$ for some rational $v > w$, and use Lemma 3.1). So, the heart $\mathcal{A}_w$ of the corresponding t-structure in $\mathcal{D}(\text{Mod-}\Lambda)$ is equivalent to the category category $\operatorname{Qcoh}\mathbb{Y}$ of quasi-coherent sheaves over a noncommutative curve of genus zero $\mathbb{Y}$.

We want to lead this to a contradiction. To this end, we investigate the category $\mathcal{H}_0$ of finite length objects in $\mathcal{H} = \operatorname{fp}(\mathcal{A}_w) \sim \operatorname{coh}\mathbb{Y}$. We know e.g. from [34, 10.1] that there is a family of connected uniserial Hom-orthogonal length categories $\mathcal{U}_y, y \in \mathbb{Y}$, such that all $\mathcal{U}_y$ have finite τ-period and $\mathcal{H}_0 = \bigcup_{y\in\mathbb{Y}}\mathcal{U}_y$. So, if S is a simple object in $\mathcal{H}_0$, then its injective envelope $E(S)$ has only finitely many non-isomorphic composition factors. Note that S is of the form $S = Y[1]$ with $Y \in \mathbf{p}_w$ or $S = Q$ with $Q \in \mathbf{q}_w$. In the first case, $Y \in \mathbf{p}_\alpha$ for some $\alpha < w$, and there is $\alpha < \beta < w$ such that $E(S)$ has all composition factors in $\mathbf{p}_\beta[1]$ and therefore $\operatorname{Hom}_\mathcal{A}(\mathbf{t}_\beta[1], E(S)) = 0$. On the other hand, Y is cogenerated by $\mathbf{t}_\beta$, so there is a non-zero map $Y \to B$ for some indecomposable module $B \in \mathbf{t}_\beta$, yielding a monomorphism $S \to B[1]$ and thus a non-zero map $B[1] \to E(S)$ in $\mathcal{A}_w$, a contradiction. This shows that the simple objects in $\mathcal{H}_0$ are all of the form $S = Q$ with $Q \in \mathbf{q}_w$, so they belong to the torsion-free class of the torsion pair $(\mathcal{C}_w[1], \mathcal{Q}_w)$ in $\mathcal{A}_w$. But then the noetherian tilting object $V = \Lambda[1] \in \mathcal{C}_w[1]$ cannot have a simple quotient, again a contradiction. □

In order to determine the pure-injectives of slope 0 or ∞, we need to investigate the non-stable tubular families $\mathbf{t}_0$ and $\mathbf{t}_\infty$. It will be convenient to work with sheaves rather than with modules. We fix a canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ of mod-Λ. The corresponding torsion pair $(\operatorname{Gen}\mathbf{q}, \mathcal{C})$ induces a t-structure whose heart $\mathcal{A}$ can be identified with the category $\operatorname{Qcoh}\mathbb{X}$ of quasi-coherent sheaves over a noncommutative curve $\mathbb{X}$ of genus zero and tubular type, cf. Section 3.1.

We know from Section 2.5 that $\operatorname{Qcoh}\mathbb{X}$ admits a coherent tilting sheaf V with endomorphism ring Λ yielding an equivalence $H_V = \operatorname{Hom}_\mathcal{A}(V, -) : \operatorname{Gen} V \to \mathcal{C}$. We now proceed as in [32, Section 4.9], where more details can be found. Let $V_1, \ldots, V_m$ be the indecomposable direct summands of V having maximal slope α as sheaves in $\operatorname{Qcoh}\mathbb{X}$. Under the functor H_V, they correspond to the indecomposable projective modules contained in the non-stable family $\mathbf{t}_0$. We write $V = V_0 \oplus V_{max}$ where $V_{max} = \bigoplus_{i=1}^m V_i$, and we denote by $\Lambda_0 = \Lambda/\Lambda e\Lambda \cong \operatorname{End}_\mathcal{A} V_0$ the algebra induced by the idempotent $e \in \Lambda$ corresponding to the direct summand V_{max}. The sheaves $V_1, \ldots, V_m$ are arranged in a union $\mathfrak{W}$ of disjoint wings inside the stable tubular family $\mathbf{t}_\alpha$ in $\operatorname{Qcoh}\mathbb{X}$ consisting of all coherent sheaves of slope α. There are precisely m rays starting (and m corays ending) in $\mathfrak{W}$.

Now a sheaf $X \in \operatorname{Qcoh}\mathbb{X}$ of slope α corresponds to a Λ-module in $\mathcal{C}$ under the functor H_V if and only if $\operatorname{Ext}^1_\mathcal{A}(V, X) = 0$. By Serre duality and slope arguments this amounts to $\operatorname{Hom}_\mathcal{A}(X, \tau V) = \operatorname{Hom}_\mathcal{A}(X, \tau V_{max}) = 0$, that is, $\operatorname{Ext}^1_\mathcal{A}(V_{max}, X) = 0$. If additionally $\operatorname{Hom}_\mathcal{A}(V_{max}, X) = 0$, then $H_V(X) = \operatorname{Hom}_\mathcal{A}(V_0, X)$ is a Λ_0-module. Notice that the algebra Λ_0 is tame concealed, its preprojective component agrees with $\mathbf{p}_0$, and its (stable) tubular family is obtained from $\mathbf{t}_\alpha$ by removing the m rays starting in $\mathfrak{W}$ and the m corays ending in $\tau\mathfrak{W}$. Moreover, there is a homological ring epimorphism $\lambda : \Lambda \to \Lambda_0$.

Let us turn to the Prüfer sheaves of slope α. We claim that they correspond to indecomposable pure-injective Λ-modules of slope 0. For a proof, we switch to the category $\operatorname{Qcoh}\mathbb{X}_\alpha$ whose indecomposable finite length objects are given by the tubular family $\mathbf{t}_\alpha$ (here $\mathbb{X}_\alpha$ is a noncommutative curve of tubular type, which is isomorphic to $\mathbb{X}$ in case k is algebraically closed), and we apply Lemma 3.4. More precisely, a Prüfer sheaf X of slope α is injective when viewed inside $\operatorname{Qcoh}\mathbb{X}_\alpha$, cf. Section 3.1 or [3, Prop. 3.6]. So X is a pure-injective sheaf in $\operatorname{Qcoh}\mathbb{X}$. Further, $\operatorname{Ext}^1_\mathcal{A}(V, X) = 0$, and the functor H_V, which preserves direct limits and direct products, maps X to a pure-injective Λ-module. This module has slope 0 because the sheaves in $\mathbf{t}_\alpha$ correspond

to Λ-modules in $\mathbf{t}_0$ except for the m corays ending in $\tau\mathfrak{W}$. We denote by $X_1, \dots, X_m$ the Λ-modules which correspond to Prüfer sheaves originating in the m rays from $\mathfrak{W}$; the remaining Prüfer sheaves correspond to the Prüfer Λ_0-modules.
Similarly, the adic sheaves of slope α correspond to Λ-modules if and only if they don't arise from corays ending in $\tau\mathfrak{W}$, in which case they are indecomposable pure-injective Λ-modules and agree with the adic modules over Λ_0. Moreover, they have slope 0 because $\mathcal{M}_0$ is closed under inverse limits by [11, Lemma 9.10].
The case $w = \infty$ is obtained by duality, since the role of 0 and ∞ is swapped when turning to left modules via the duality $D = \mathrm{Hom}_k(-, k)$. So, there is a tame concealed factor algebra Λ_∞ of Λ whose preinjective component agrees with $\mathbf{q}_\infty$, and there are indecomposable pure-injective Λ-modules of slope ∞ which, except from, say, ℓ modules denoted by $Y_1, \dots, Y_\ell$, are precisely the adic Λ_∞-modules. Furthermore, also the Prüfer modules over Λ_∞ are pure-injective Λ-modules.
Finally, for both $w = 0$ and $w = \infty$, the generic module over Λ_w occurs as a direct summand in a direct product of copies of any Prüfer Λ_w-module, and so it is a pure-injective Λ-module of slope w.

Let us now summarize our findings.

Theorem 6.7. *The following is a complete list of the indecomposable pure-injective Λ-modules:*

1. *the finite dimensional indecomposable modules,*
2. *the Prüfer modules, the adic modules, and the generic module of slope w with $w \in \mathbb{Q}^+$,*
3. *the indecomposable modules in* $\mathrm{Prod}\,\mathbf{W}_w$ *with $w \in \mathbb{R}^+ \setminus \mathbb{Q}^+$,*
4. *the Prüfer modules, the adic modules, and the generic module over Λ_0 and Λ_∞,*
5. *the modules $X_1, \dots, X_m$ and $Y_1, \dots, Y_\ell$ defined above.*

Proof. By the discussion above, all modules in the list are indecomposable pure-injective, so we only have to show that the list is complete. Every infinite dimensional indecomposable pure-injective module has a slope w by [39, 13.1]. Combining [27, Lemma 50] with Corollary 6.6, we get the statement for $w \in \mathbb{R}^+$.
Let now $w \in \{0, \infty\}$. We discuss the case $w = 0$, the case $w = \infty$ is obtained by duality. By Proposition 3.5 we can assume that our module is not a Λ_0-module. Keeping the notation as above, it then has the form $H_V(I)$ for an indecomposable sheaf $I \in \mathrm{Qcoh}\,\mathbb{X}$ with $\mathrm{Hom}_{\mathcal{A}}(\mathfrak{W}, I) \neq 0$. Consider the canonical exact sequence

$$0 \to t(I) \to I \to I/t(I) \to 0$$

induced by the torsion pair $(\mathrm{Gen}\,\mathfrak{t}_\alpha, \mathfrak{F}_\alpha)$ generated by $\mathfrak{t}_\alpha$ in $\mathrm{Qcoh}\,\mathbb{X}$. Observe first that $t(I)$ cannot vanish because I is not torsionfree by our assumption. By [3, Prop. 3.7], the sheaf $t(I)$ then has an indecomposable pure-injective summand which is either coherent or a Prüfer sheaf. Since the sequence is pure-exact by [3, Rem. 3.3], this summand must coincide with the indecomposable sheaf I, and our module $H_V(I)$ must then be isomorphic to one of $X_1, \dots, X_m$. □

Remark 1. (1) Every module in $\mathcal{C}_w$, $w \in \mathbb{R}^+ \cup \{0\}$, has projective dimension at most one, and every module in $\mathcal{B}_w$, $w \in \mathbb{R}^+ \cup \{\infty\}$, has injective dimension at most one, as a consequence of Lemma 3.1. In particular, all modules in $\mathbf{p}_\infty$ have projective dimension at most one, and all modules in $\mathbf{q}_0$ have injective dimension at most one.

(2) Let $w \in \{0, \infty\}$. There are a cotilting module $\mathbf{W}_w$ and a tilting module $\mathbf{L}_w$ of slope w such that $\mathcal{C}_w = {}^\perp\mathbf{q}_w = \mathrm{Cogen}\,\mathbf{W}_w$ and $\mathcal{B}_w = {\mathbf{p}_w}^\perp = \mathrm{Gen}\,\mathbf{L}_w$. This is a consequence of Theorem 2.1 and (1). That $\mathbf{W}_w$ and $\mathbf{L}_w$ have slope w is shown as in the proof of Theorem 6.4.

(3) Up to equivalence, $\mathbf{W}_0 = \lambda_*(W) \oplus X_1 \oplus \dots \oplus X_m$, where W is the direct sum of the generic and all Prüfer modules over Λ_0 and $\lambda_* : \text{Mod-}\Lambda_0 \to \text{Mod-}\Lambda$ is the embedding given by the restriction of scalars along the ring epimorphism $\lambda : \Lambda \to \Lambda_0$.
Indeed, recall first that $\mathbf{W}_0 = H_V(I)$ where I is the sum of the generic and the Prüfer sheaves of slope α in $\mathrm{Qcoh}\,\mathbb{X}$ (with α as above). We switch again to $\mathrm{Qcoh}\,\mathbb{X}_\alpha$, which can be viewed as the heart of the faithful torsion pair in $\mathrm{Qcoh}\,\mathbb{X}$ generated by the class $\mathfrak{q}_\alpha$ of coherent sheaves of slope

$> \alpha$. Now I corresponds to an injective cogenerator of $\operatorname{Qcoh}\mathbb{X}_\alpha$, so arguing as in Subsection 2.3, we see that I satisfies conditions (C1)-(C3) and $\operatorname{Cogen} I = {}^\perp I = \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(\mathfrak{q}_\alpha, -)$.
Next, we turn to our fixed canonical trisection $(\mathbf{p}, \mathbf{t}, \mathbf{q})$ of mod-Λ with the induced split torsion pair $(\operatorname{Gen}\mathbf{q}, \mathcal{C})$ in Mod-Λ and tilted torsion pair $(\operatorname{Gen} V, \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(V, -))$ in $\mathcal{A} = \operatorname{Qcoh}\mathbb{X}$. The torsion class $\operatorname{Gen} V = V^\perp$ contains $\mathfrak{q}_\alpha$ by slope arguments. This implies that $\operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(\mathfrak{q}_\alpha, -)$ contains the corresponding torsionfree class $\operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(V, -)$, and every module $Q \in \operatorname{Gen}\mathbf{q}$ satisfies $\operatorname{Ext}^1_{\mathcal{A}}(Q, I) = 0$.
Now we are ready to prove our claim. As in Lemma 2.3, we see that $\mathbf{W}_0$ verifies conditions (C2) and (C3). Of course, $\operatorname{Cogen}\mathbf{W}_0 \subset \mathcal{C}_0$. Further, $\mathcal{C}_0 \subset {}^\perp\mathbf{W}_0$, because any module $C \in \mathcal{C}_0$ belongs to $\mathcal{C}$ and corresponds to a sheaf in $\operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(\mathfrak{q}_\alpha, -)$ and thus satisfies $\operatorname{Ext}^1_\Lambda(C, \mathbf{W}_0) \cong \operatorname{Ext}^1_{\mathcal{A}}(C[1], I) = 0$ by Lemma 2.2(d). Let us check ${}^\perp\mathbf{W}_0 \subset \operatorname{Cogen}\mathbf{W}_0$. Take $M \in {}^\perp\mathbf{W}_0$, denote by $M' = \operatorname{Rej}_{\mathbf{W}_0}(M)$ the intersection of all kernels of homomorphisms $M \to \mathbf{W}_0$, and set $\overline{M} = M/M'$. Then $\overline{M} \in \operatorname{Cogen}\mathbf{W}_0$ belongs to ${}^\perp\mathbf{W}_0$ and to $\mathcal{C}_0$, and in particular it has projective dimension at most one by (1). So $\operatorname{Ext}^1_\Lambda(\overline{M}, \mathbf{W}_0) = \operatorname{Ext}^2_\Lambda(\overline{M}, \mathbf{W}_0) = 0$. Applying the functor $\operatorname{Hom}_\Lambda(-, \mathbf{W}_0)$ to the exact sequence $0 \to M' \to M \to \overline{M} \to 0$ thus yields a long exact sequence $0 \to \operatorname{Hom}_\Lambda(\overline{M}, \mathbf{W}_0) \to \operatorname{Hom}_\Lambda(M, \mathbf{W}_0) \to \operatorname{Hom}_\Lambda(M', \mathbf{W}_0) \to \operatorname{Ext}^1_\Lambda(\overline{M}, \mathbf{W}_0) \to \operatorname{Ext}^1_\Lambda(M, \mathbf{W}_0) \to \operatorname{Ext}^1_\Lambda(M', \mathbf{W}_0) \to 0$ where the first map is an isomorphism by construction, and the fourth and fifth term vanish. Then $\operatorname{Hom}_\Lambda(M', \mathbf{W}_0) = \operatorname{Ext}^1_\Lambda(M', \mathbf{W}_0) = 0$, which implies $M' = 0$ by condition (C3), and proves that $M \in \operatorname{Cogen}\mathbf{W}_0$. We conclude that $\mathbf{W}_0$ is a cotilting Λ-module cogenerating $\mathcal{C}_0$.

6.5. Tilting modules and sheaves. We now turn to the classification of tilting modules. Let us summarize our findings in Sections 6.2 and 6.3.

Corollary 6.8. *The tilting and cotilting modules of slope $w \in \mathbb{Q}^+$ are parametrized by the pairs (Y, P) given by a branch object Y in $\mathbf{t}_w$ and a subset $P \subset \mathbb{X}_w$, where $\mathbb{X}_w$ is a noncommutative curve of genus zero (again tubular and derived equivalent to $\mathbb{X}$ by* [33, 8.1.6]*) parametrizing the family $\mathbf{t}_w$. The tilting and cotilting modules of irrational slope w are equivalent to $\mathbf{L}_w$ and $\mathbf{W}_w$, respectively.*

We are going to see that all tilting modules in the "central part" of Mod-Λ, that is, contained in a smallest $\mathcal{C}_w$ with $0 < w < \infty$, arise in this way. This will enable us to recover the classification of the tilting sheaves over a noncommutative curve of genus zero of tubular type from [3]. We first need a preliminary result.

Lemma 6.9. *Let T be a large tilting module with $\mathcal{S} = {}^\perp(T^\perp) \cap \text{mod-}\Lambda$.*
(1) *Let $w \in \mathbb{R}^+ \cup \{0, \infty\}$. Then $T \in \mathcal{C}_w$ if and only if $\mathcal{S} \subset \mathcal{C}_w$.*
(2) *If $T \in \mathcal{C}_w$ with $w \in \mathbb{Q}^+$, and $\mathcal{S} \cap \mathbf{t}_w \neq \emptyset$, then T has slope w.*

Proof. (1) is shown as in Lemma 5.4(1).
(2) If $\mathcal{S}$ contains a ray from $\mathbf{t}_w$, then we know from Lemma 5.4 that $\operatorname{Add} T$ contains a Prüfer module S_∞ of slope w. Then $\operatorname{Ext}^1_\Lambda(S_\infty, T) = 0$, and using that S_∞ has projective dimension at most one, it follows from Lemma 3.3 that T cannot have non-zero factor modules in $\mathcal{P}_w$. Considering the torsion pair $(\mathcal{B}_w, \mathcal{P}_w)$, we infer that $T \in \mathcal{B}_w$, so T has slope w.
If $\mathcal{S}$ does not contain a complete ray from $\mathbf{t}_w$, then we know from Lemma 5.4 and Proposition 5.1 that T has the form $T = Y \oplus M$ where $0 \neq Y \in \operatorname{add}\mathbf{t}_w$ and $M \in \mathcal{F}_w$. Let $\mathcal{U}$ be the set of regular composition factors of Y. Then $\Lambda_{\mathcal{U}}$ is a concealed canonical algebra of domestic type (cf. [3, Sec. 2]), and by Proposition 5.2, M is a large torsion-free tilting module over $\Lambda_{\mathcal{U}}$. By the classification in Theorem 6.2 it follows that M is equivalent to the Lukas tilting module over $\Lambda_{\mathcal{U}}$. In particular $\operatorname{Hom}_{\Lambda_{\mathcal{U}}}(M, P') = 0$ for all $P' \in \mathbf{p}_{\mathcal{U}}$. Now every $P \in \mathbf{p}_w$ embeds in $P \otimes_\Lambda \Lambda_{\mathcal{U}} \in \mathbf{p}_{\mathcal{U}}$ by Proposition 4.4, thus also $\operatorname{Hom}_\Lambda(M, P) = 0$. We conclude that $M \in \mathcal{B}_w$, and T has slope w. $\square$

Theorem 6.10. *Let T be a large tilting module, and assume there is $w \in \mathbb{R}^+$ such that $T \in \mathcal{C}_w$ but $T \notin \mathcal{C}_\alpha$ for any $\alpha < w$. Then T has slope w.*

Proof. First of all, we know from Lemma 6.9 that $\mathcal{S} = {}^{\perp}(T^{\perp}) \cap \text{mod-}\Lambda \subset \mathcal{C}_w$, and we can assume w.l.o.g. $\mathcal{S} \subset \operatorname{add} \mathbf{p}_w$. This is clear if w is irrational, because $\mathcal{C}_w \cap \text{mod-}\Lambda = \operatorname{add} \mathbf{p}_w$, and for $w \in \mathbb{Q}^+ \cup \{\infty\}$ it follows from Lemma 6.9(2).
Furthermore, the assumption on w implies that $\mathcal{S}$ is not contained in $\operatorname{add} \mathbf{p}_\alpha$ for any $\alpha < w$. So there is an increasing sequence of rational numbers $\alpha_1 < \alpha_2 < \ldots < w$ converging to w such that $\mathcal{S} \cap \mathbf{t}_{\alpha_i} \neq \emptyset$ for all i. Let us consider the increasing sequence of subcategories

$$\mathcal{S}_1 \subset \mathcal{S}_2 \ldots \subset \text{mod-}\Lambda$$

given by $\mathcal{S}_i = \mathcal{S} \cap \mathcal{C}_{\alpha_i}$ for all $i \in \mathbb{N}$. Since $\mathcal{S}$ and $\mathcal{C}_{\alpha_i} = {}^{\perp}\mathbf{q}_{\alpha_i}$ are resolving subcategories, the $\mathcal{S}_i$ are resolving subcategories of mod-Λ giving rise to a decreasing sequence of tilting classes

$$\mathcal{S}_1{}^{\perp} \supset \mathcal{S}_2{}^{\perp} \supset \ldots$$

For each $i \in \mathbb{N}$ let T_i be a tilting module with $T_i{}^{\perp} = \mathcal{S}_i{}^{\perp}$. By the bijection in Theorem 2.1 we have $\mathcal{S}_i = {}^{\perp}(T_i^{\perp}) \cap \text{mod-}\Lambda$, and since $\mathcal{S}_i \cap \mathbf{t}_{\alpha_i} \neq \emptyset$, we infer from Lemma 6.9 that T_i has slope α_i.
Next we observe that $\mathcal{S} = \bigcup_{i \in \mathbb{N}} \mathcal{S}_i$. Indeed, if $X \in \mathcal{S}$, and X is indecomposable w.l.o.g., then $X \in \mathbf{p}_w$ either belongs to $\mathbf{p}_0$ and is therefore contained in all $\mathcal{S}_i$, or it belongs to $\mathbf{t}_\alpha$ for some $\alpha < w$ and is therefore contained in $\mathcal{S}_i = \mathcal{S} \cap \mathcal{C}_{\alpha_i}$ for i with $\alpha < \alpha_i < w$.
It follows that $\mathcal{S}^{\perp} = \bigcap_{i \in \mathbb{N}} \mathcal{S}_i{}^{\perp} = \bigcap_{i \in \mathbb{N}} T_i{}^{\perp}$ and thus $T \in \bigcap_{i \in \mathbb{N}} T_i{}^{\perp}$. We claim that $T \in \mathcal{B}_w$, which will yield that T has slope w. By Lemma 6.3, the claim amounts to showing that $T \in \mathcal{B}_v$ for all $v < w$. So take $v < w$ and $i \in \mathbb{N}$ such that $v < \alpha_i < w$. Observe that T_i belongs to $\mathcal{B}_{\alpha_i} \subset \mathcal{B}_v$ because it has slope α_i. As $\mathcal{B}_v$ is a torsion class, also $T_i{}^{\perp} = \operatorname{Gen} T_i \subset \mathcal{B}_v$. But then also $T \in T_i{}^{\perp}$ belongs to $\mathcal{B}_v$, which completes the proof. □

Remark 2. By [13] one can realize a tilting module T as above as a direct limit of tilting modules of increasing slope $\alpha_1 < \alpha_2 < \ldots < w$.

Furthermore, Theorem 6.10 has a dual version: if C is a large cotilting module, and there is $w \in \mathbb{R}^+$ such that $C \in \mathcal{B}_w$ but $C \notin \mathcal{B}_\alpha$ for any $\alpha > w$, then C has slope w. This is proved by using that the dual module $D(C)$ is a tilting module in the central part of Λ-Mod.

Let now $\mathbb{X}$ be a noncommutative curve of genus zero of tubular type. The slope of an indecomposable coherent sheaf is defined as the ratio of the degree by the rank. It is a rational number, unless the sheaf has finite length, in which case the rank is zero and the slope ∞. The coherent sheaves of a given slope $w \in \mathbb{Q} \cup \{\infty\}$ form a tubular family denoted by $\widehat{\mathbf{t}}_w$. One then extends the notion of slope to all quasi-coherent sheaves like in Section 6.1. For details, we refer to [3]. We can now recover the classification of large quasi-coherent tilting sheaves over $\mathbb{X}$ from [3, Thm. 7.14].

Corollary 6.11. *Let $\mathbb{X}$ be a noncommutative curve of genus zero of tubular type. Then every large tilting sheaf in* $\operatorname{Qcoh} \mathbb{X}$ *has a slope $w \in \mathbb{R} \cup \{\infty\}$. The large tilting sheaves of slope $w \in \mathbb{Q} \cup \{\infty\}$ are parametrized by the pairs (Y, P) given by a branch object Y in $\widehat{\mathbf{t}}_w$ and a subset $P \subset \mathbb{X}_w$, where $\mathbb{X}_w$ is a noncommutative curve of genus zero parametrizing the family $\widehat{\mathbf{t}}_w$. The tilting sheaves of irrational slope w are equivalent to the Lukas tilting sheaf $\widehat{\mathbf{L}_w}$.*

Proof. For every tilting object $\widehat{T}$ in $\mathcal{A} = \operatorname{Qcoh} \mathbb{X}$ one can find a tilting bundle T_{cc} in $\operatorname{coh} \mathbb{X}$ such that $\widehat{T} \in \operatorname{Gen} T_{cc}$, cf. [3, Lem. 7.10]. Then $\Lambda = \operatorname{End}_{\mathcal{A}} T_{cc}$ is a concealed canonical tubular algebra derived equivalent to $\mathcal{A}$, and Mod-Λ can be viewed as the heart of the torsion pair $(\operatorname{Gen} T_{cc}, \operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(T_{cc}, -))$ in $\mathcal{A}$. Under the tilting functor $\operatorname{Hom}_{\mathcal{A}}(T_{cc}, -)$, the tubular family in $\operatorname{coh} \mathbb{X}$ formed by the indecomposable sheaves of finite length becomes a tubular family $\mathbf{t}_\alpha$ in the AR-quiver of Λ. Since $\operatorname{coh} \mathbb{X}$ has neither projective nor injective objects, $\mathbf{t}_\alpha$ is stable, that is, $\alpha \in \mathbb{Q}^+$. The torsion pair $(\operatorname{Gen} T_{cc}, \operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(T_{cc}, -))$ can now be viewed as the tilt of the torsion pair $(\mathcal{Q}_\alpha, \mathcal{C}_\alpha)$ in Mod-Λ, that is, $\operatorname{Gen} T_{cc} = \mathcal{C}_\alpha[1]$ and $\operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(T_{cc}, -) = \mathcal{Q}$, compare Sections 2.5 and 3.1. We infer from Corollary 2.6 that $\widehat{T} = T[1]$ for a tilting Λ-module $T \in \mathcal{C}_\alpha$, and T must have slope $w \leq \alpha$ by Theorem 6.10. The statement now follows from Corollary 6.8. Here $\widehat{\mathbf{L}_w} = \mathbf{L}_w[1]$. □

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